$$V^{\pi}(s) = \mathbb{E}_{\pi}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \,\middle|\, s_1 = s \right]$$

π deterministic.

$$= \mathbb{E}_{\pi}\left[r_1 + \sum_{t=2}^{\infty} \gamma^{t-1} r_t \,\middle|\, s_1 = s \right].$$

$$= \mathbb{E}_{\pi}[r_1 \mid s_1 = s] + \gamma\, \mathbb{E}_{\pi}\left[\sum_{t=2}^{\infty} \gamma^{t-2} r_t \,\middle|\, s_1 = s \right].$$

$$= R(s, \pi(s)) + \gamma\, \mathbb{E}_{s_2 \mid s_1}\Bigg[\underbrace{\mathbb{E}_{\pi}\left[\sum_{t=2}^{\infty} \gamma^{t-2} r_t \,\middle|\, s_1 = s,\ s_2 = s' \right]}_{fn(s,\,s')} \Bigg].$$

$$= R(s, \pi(s)) + \gamma \sum_{s'} P(s' \mid s, \pi(s))\, \mathbb{E}_{\pi}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_{t+1} \,\middle|\, s_1 = s,\ s_2 = s' \right].$$

$$\mathbb{E}_{\pi}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_{t+1} \,\middle|\, \cancel{s_1 = s},\ s_2 = s' \right]$$

$$s_1 \xrightarrow[r_1]{a_1} \boxed{s_2 \xrightarrow[r_2]{a_2} s_3 \xrightarrow[r_3]{a_3} \cdots}$$

$$\mathbb{E}_{\pi}\left[r_1 + \gamma r_2 + \gamma^2 r_3 + \cdots \,\middle|\, s_1 = s' \right].$$

$$= V^\pi(s').$$

$$V^\pi(s) = R(s,\pi(s)) + \gamma \sum_{s'} P(s' \mid s,\pi(s)) \cdot V^\pi(s').$$

$I - \gamma P^\pi$ is inv.

$$\boxed{\begin{aligned} &[P^\pi]_{s,s'} \\ &= P^\pi(s' \mid s) \\ &= P(s' \mid s,\pi(s)) \end{aligned}}$$

$\boxed{\forall x \neq \vec{0}.}$

$$\|(I - \gamma P^\pi)x\|_\infty$$

$$\|v\|_\infty := \max_i |v_i|$$

$$= \|x - \gamma P^\pi x\|_\infty.$$

$$\geq \underbrace{\|x\|_\infty}_{>0.} - \|\gamma P^\pi x\|_\infty. \quad < \|x\|_\infty.$$

$$= \|x\|_\infty - \gamma \|P^\pi x\|_\infty. \geq (1-\gamma)\|x\|_\infty$$

It suffices to show: $\|P^\pi x\|_\infty \leq \|x\|_\infty.$

$$\Updownarrow$$

$$\forall s: \quad |\underbrace{P^\pi(\cdot \mid s)^T}_{\mathbb{R}^S} \underbrace{x}_{\mathbb{R}^S}| \leq \|x\|_\infty.$$

$$\Updownarrow$$

$$\left| \mathbb{E}_{s' \sim P^\pi(\cdot \mid s)}[x(s')] \right|. \leq \max_{s'} |x(s')|.$$