CS 542 Statistical Reinforcement Learning

Nan Jiang

What's this course about?

- A grad-level seminar course on **theory of RL**
- with focus on sample complexity analyses
- all about proofs, some perspectives, 0 implementation
- No text book; material is created by myself (course notes)
  - Related monograph under development w/ Alekh Agarwal, Sham Kakade, and Wen Sun
  - See course website for more material and references

Who should take this course?

- This course will be a good fit for you if you either
  - (A) have exposure to RL + comfortable with long mathematical derivations + interested in understanding RL from a purely theoretical perspective
  - (B) have solid grasp in a related theory field (e.g., theoretical computer science or learning theory) and are comfortable with **highly abstract** description of concepts / models / algorithms
- For people not in (A) or (B): I also teach CS443 RL (Spring), which focuses less on analyses & proofs and more on algorithms & intuitions

Prerequisites

- Maths
  - Linear algebra, probability & statistics, basic calculus
  - Markov chains
  - Optional: stochastic processes, numerical analysis
  - Useful: TCS background, empirical processes and statistical learning theory, optimization, control, information theory, game theory, online learning, etc. etc.
- Exposure to ML
  - e.g., CS 446 Machine Learning
  - Experience with RL

Coursework

- Some readings after/before class
- 3~4 graded homeworks to help digest certain material.
  - about 40% of final grades (rest is project)
- Course project (work on your own)
  - Baseline: reproduce theoretical analysis in existing papers
  - Advanced: identify an interesting/challenging extension to the paper and explore the novel research question yourself
  - Or, just work on a novel research question (must have a significant **theoretical** component; need to discuss with me)

Course project (cont.)

- See list of references and potential topics on website
  - To be updated this semester
- You will need to submit:
  - A brief proposal (~1/2 page). Tentative deadline: end of Oct
    - what's the topic and what papers you plan to work on
    - why you choose the topic: what interest you?
    - which aspect(s) you will focus on?
  - Final report: clarity, precision, and brevity are greatly valued. More details to come…
- All docs should be in pdf. Final report should be prepared using LaTeX.

Contents of the course

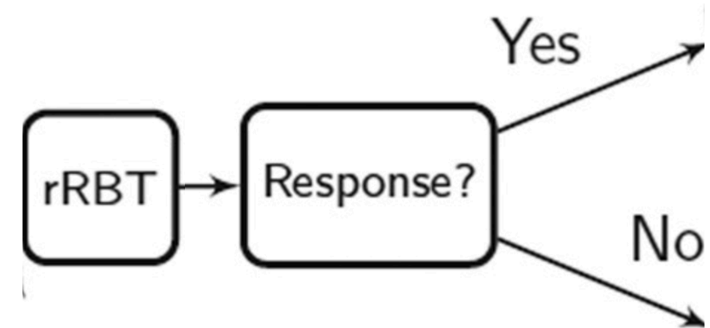
- many important topics in RL will not be covered in depth (e.g., TD). Read more (e.g., Sutton & Barto book) if you want to get a more comprehensive view of RL
- the other opportunity to learn what's not covered in lectures is the project, as potential topics for projects are much broader than what's covered in class.

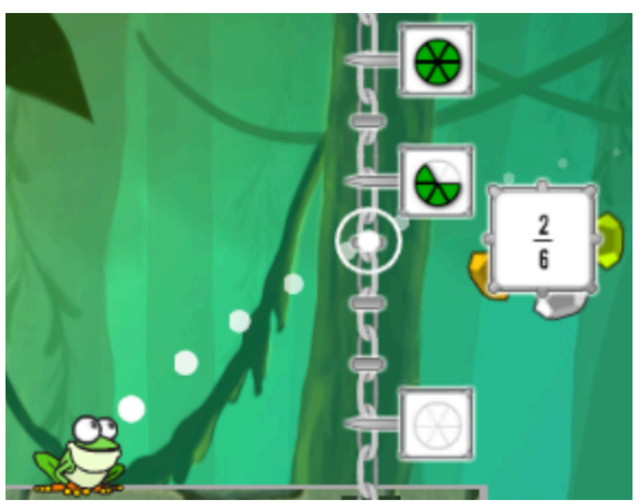
Logistics

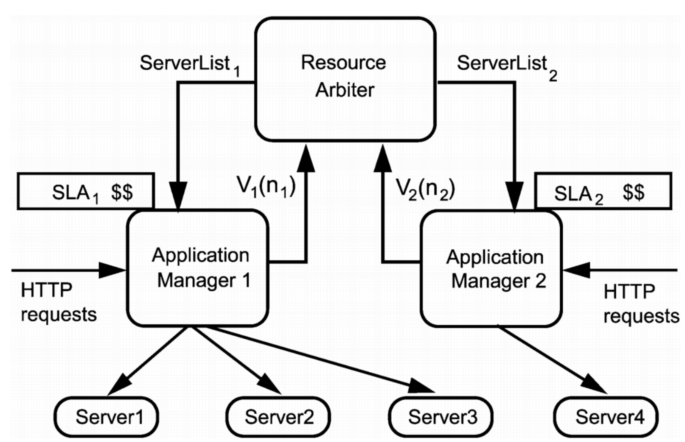
- Course website: http://nanjiang.cs.illinois.edu/cs542/
  - logistics, links to slides/notes, and resources (e.g., textbooks to consult, related courses)
- Canvas for Q&A and announcements: see link on website.
  - **Please pay attention to Canvas announcements**
  - Auditing students: please contact TA to be added to Canvas
- Recording: published on MediaSpace (link on website)
- Time: Wed & Fri 2-3:15pm.
- TA: Philip Amortila (philipa4), Yuheng Zhang (yuhengz2)
- Office hours: after lecture (TA ad hoc OH TBA)

Introduction to MDPs and RL

Reinforcement Learning (RL) Applications



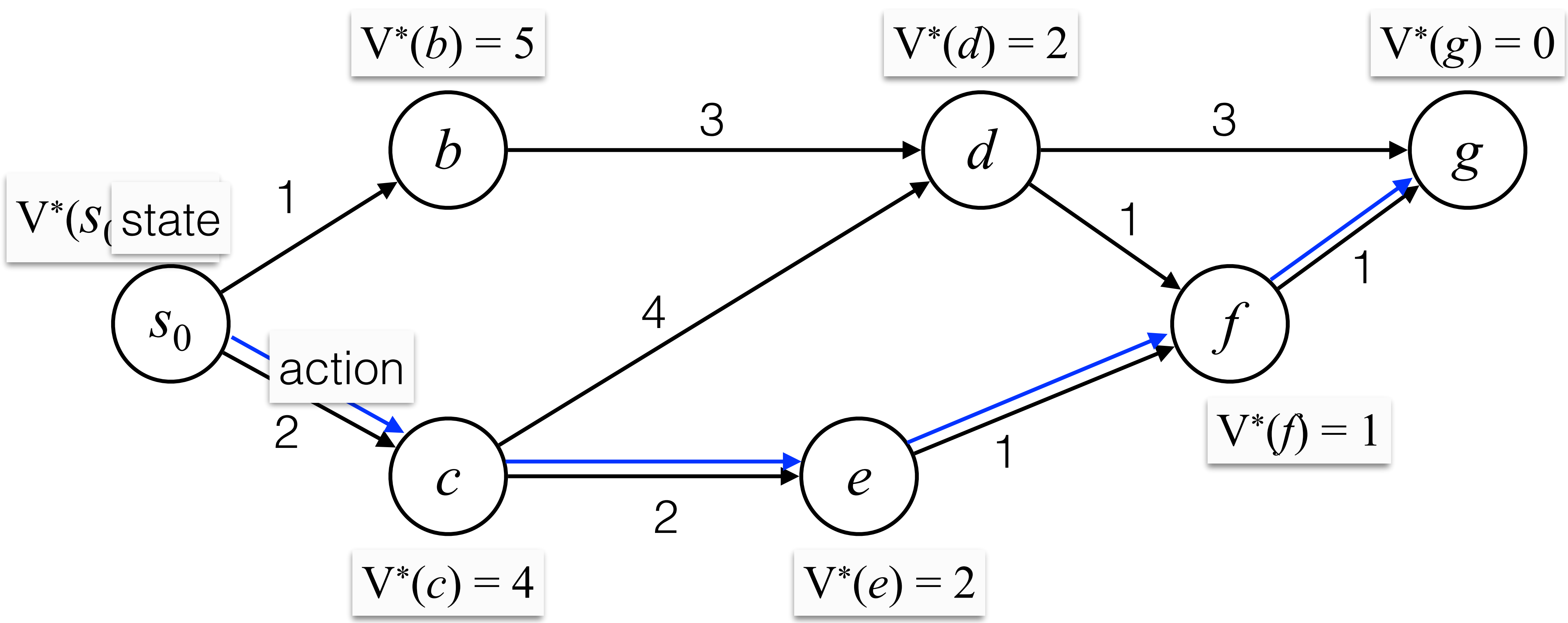




[Levine et al’16] [Ng et al’03] [Singh et al’02] [Lei et al’12]

[Mandel et al’16] [Tesauro et al’07] [Mnih et al’15][Silver et al’16]

Shortest Path

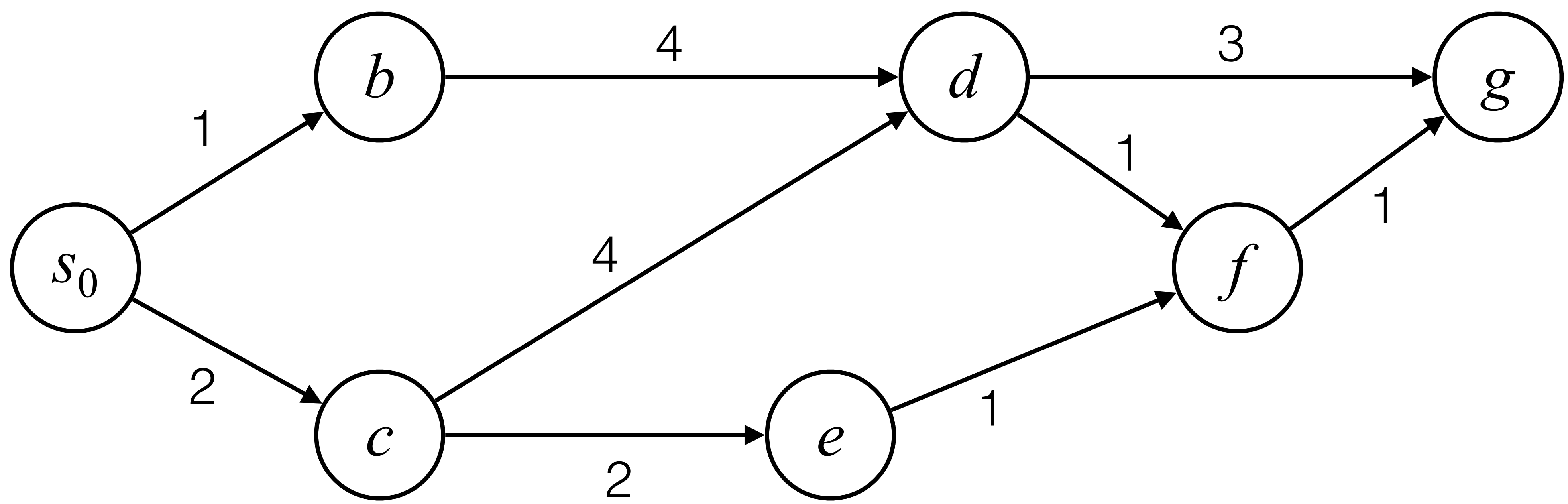


Bellman Equation $V^*(d) = \min\{3 + V^*(g)\ ,\ 1 + V^*(f)\}$

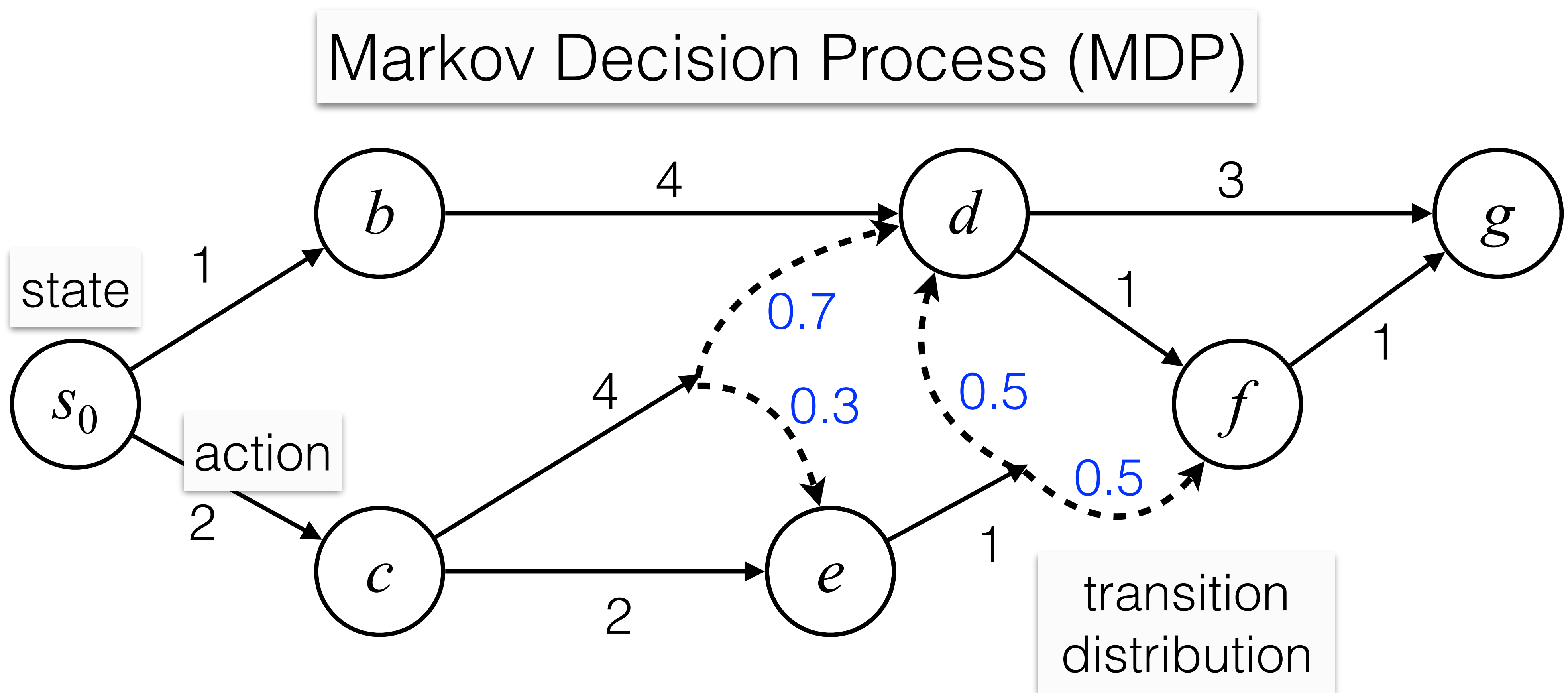
Greedy is suboptimal due to delayed effects

Need **long-term planning**

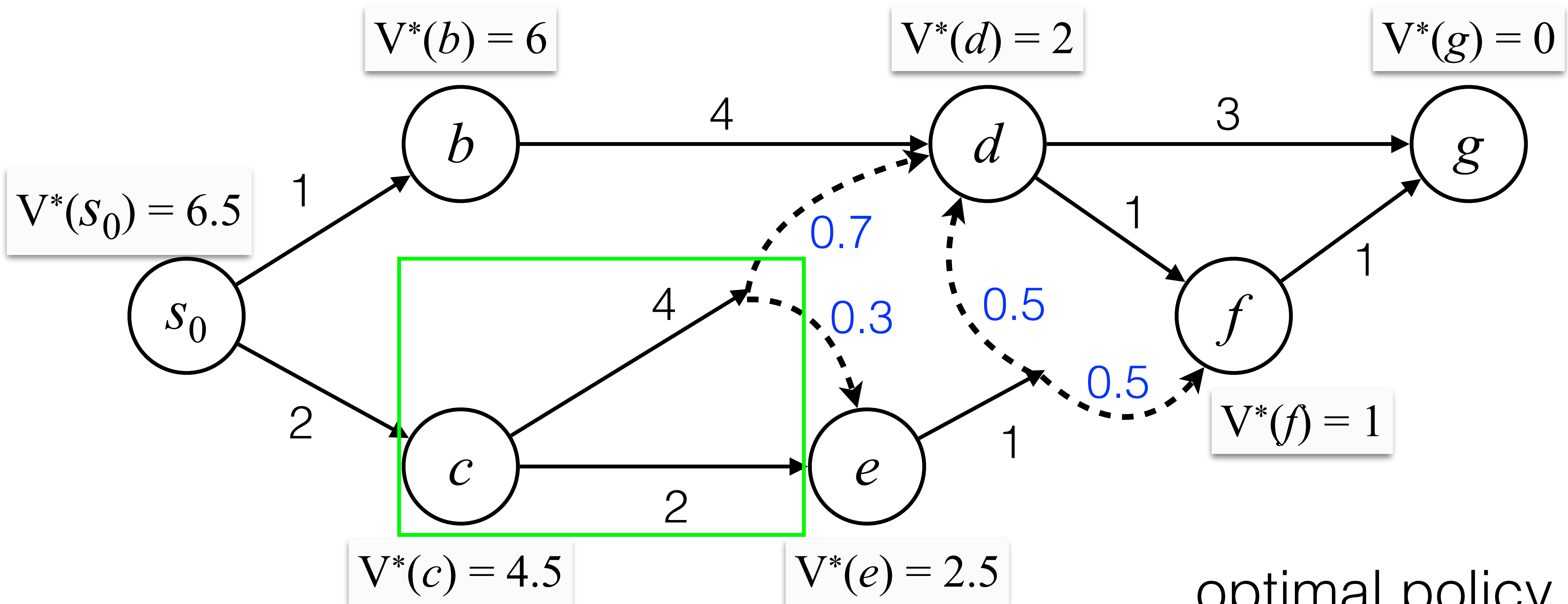
Shortest Path



Stochastic Shortest Path



Stochastic Shortest Path



optimal policy π^*

s	$\pi^*(s)$
s_0	↘
b	→
c	→
…	…

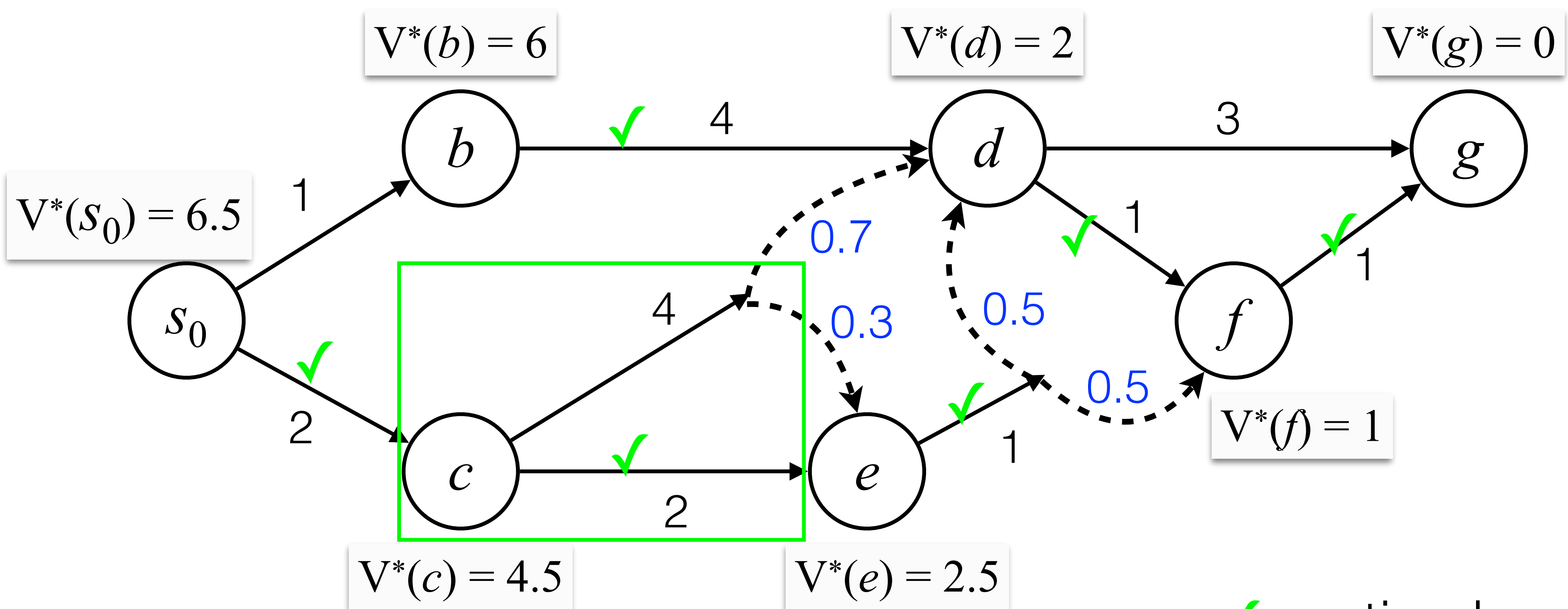
Bellman Equation

$$V^*(c) = \min\{4 + 0.7 \times V^*(d) + 0.3 \times V^*(e)\ ,\ 2 + V^*(e)\}$$

Greedy is suboptimal due to delayed effects

Need **long-term planning**

Stochastic Shortest Path



✓: optimal policy π^*

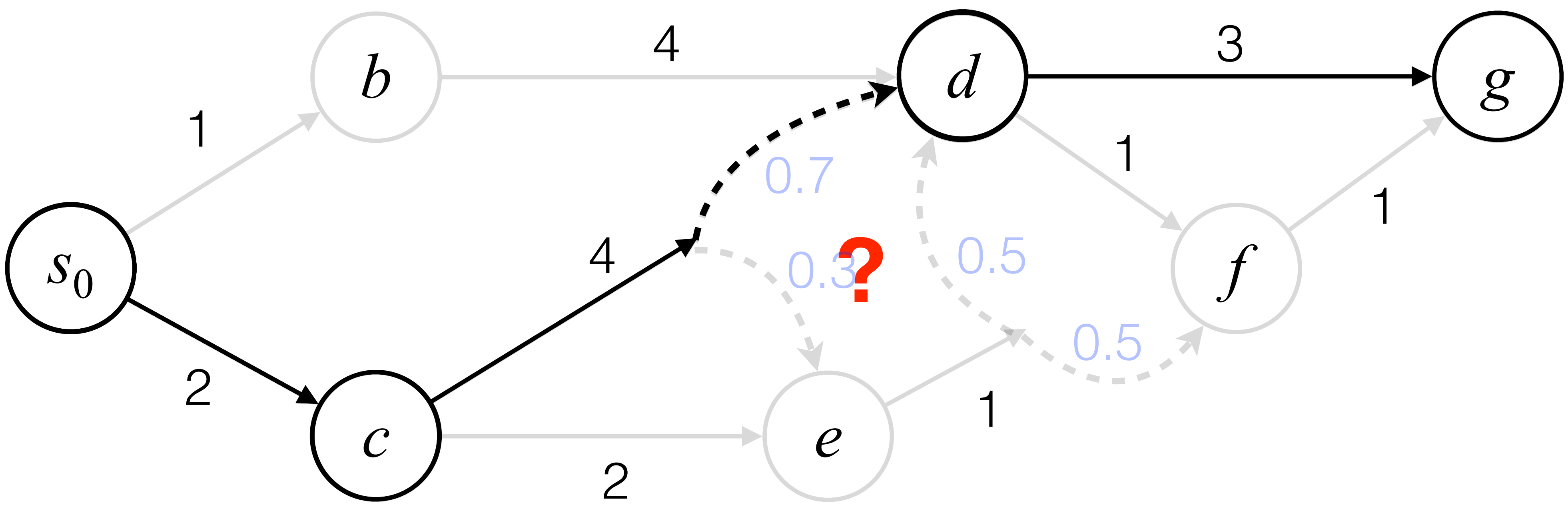
Bellman Equation

$$V^*(c) = \min\{4 + 0.7 \times V^*(d) + 0.3 \times V^*(e)\ ,\ 2 + V^*(e)\}$$

Greedy is suboptimal due to delayed effects

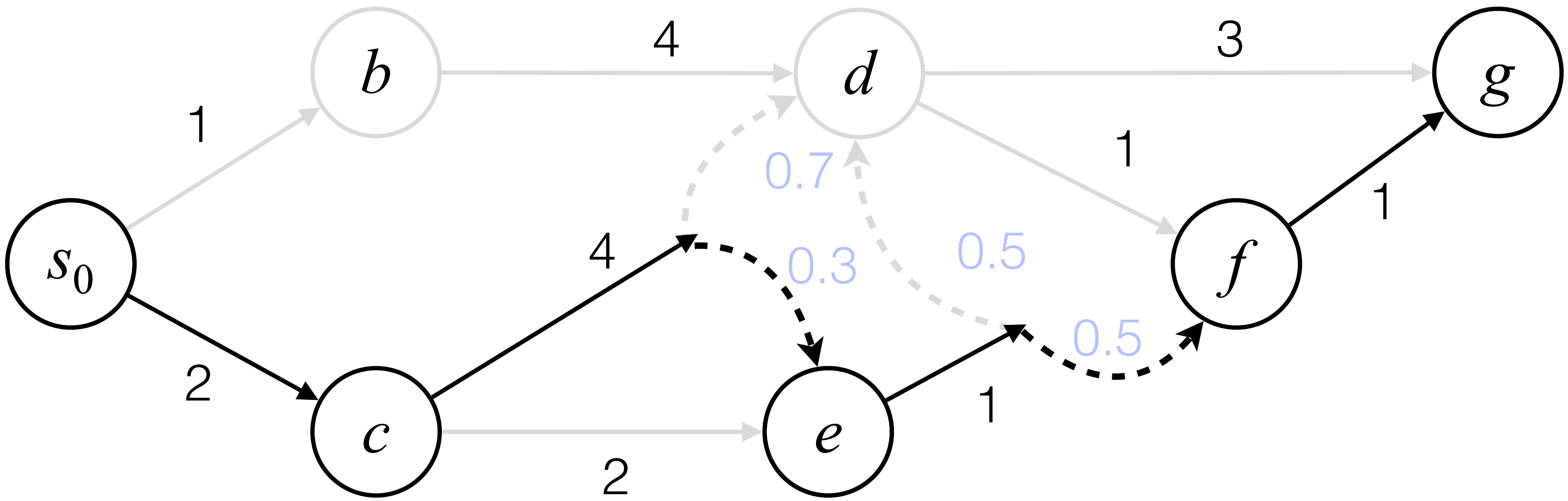
Need **long-term planning**

Stochastic Shortest Path
via trial-and-error



s_0

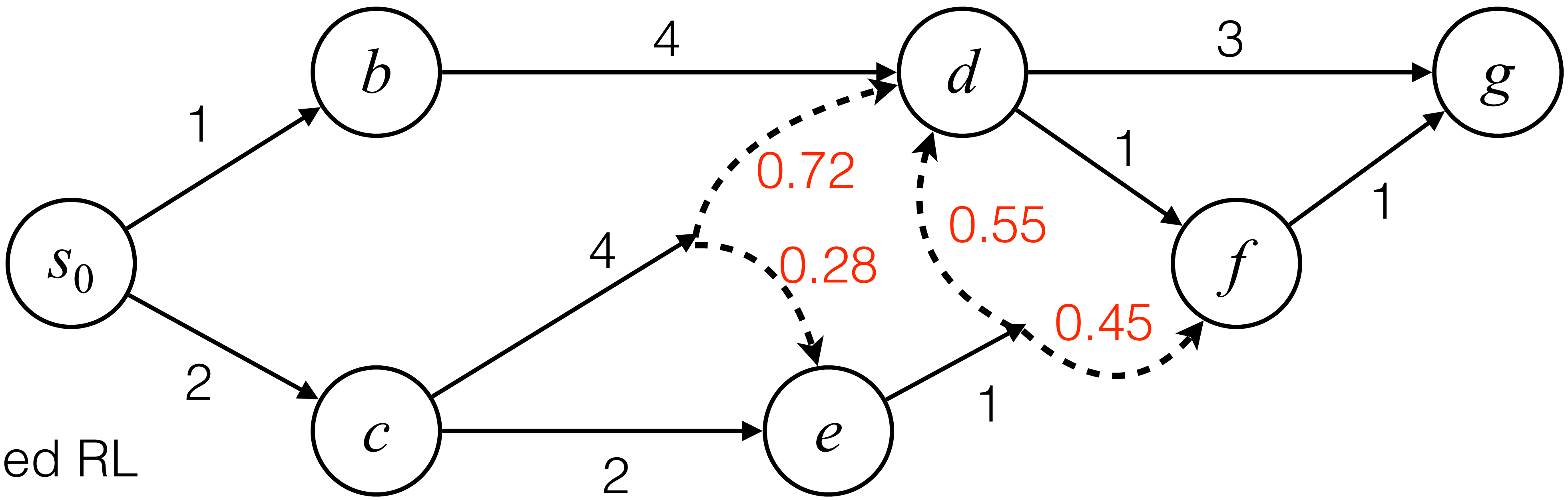
Stochastic Shortest Path
via trial-and-error



Trajectory 1: $s_0 \searrow c \nearrow d \rightarrow g$

Trajectory 2:

Stochastic Shortest Path
via trial-and-error



Model-based RL

Trajectory 1: $s_0 \searrow c \nearrow d \rightarrow g$

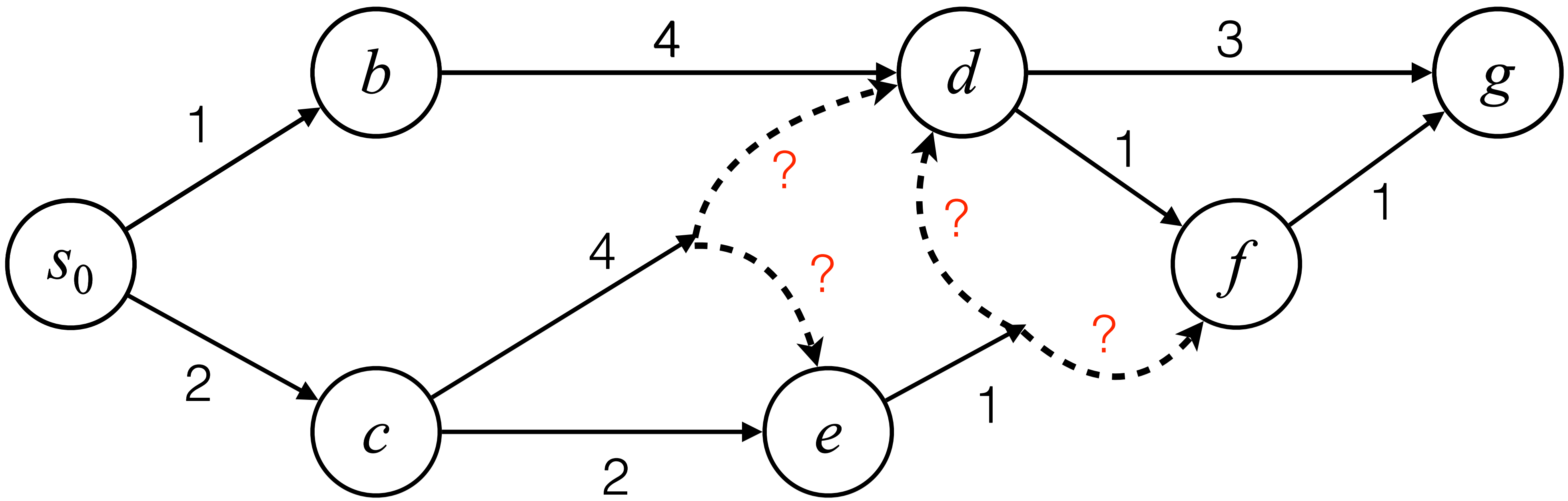
Trajectory 2: $s_0 \searrow c \nearrow e \rightarrow f \nearrow g$

…

How many trajectories do we need to compute a near-optimal policy?

sample complexity

Stochastic Shortest Path
via trial-and-error



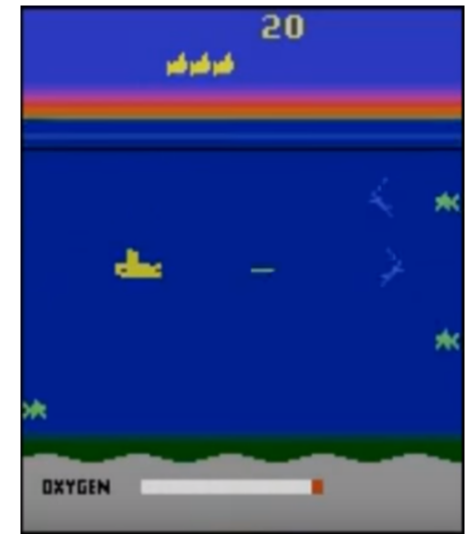
How many trajectories do we need to compute a near-optimal policy?

Nontrivial! Need **exploration**

- Assume states & actions are visited uniformly
- #trajectories needed $\leq n \cdot$ (#state-action pairs)

#samples needed to estimate a multinomial distribution

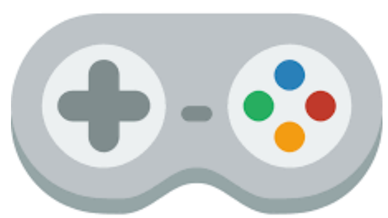
Video game playing



reward $r_t = R\,(s_t\ ,\ a_t)$

state $s_t \in S$

+20



e.g., random spawn of enemies

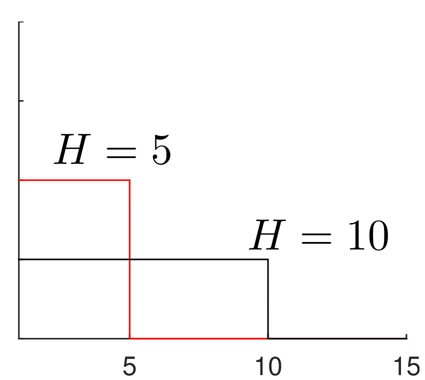
policy $\pi\colon S \rightarrow A$

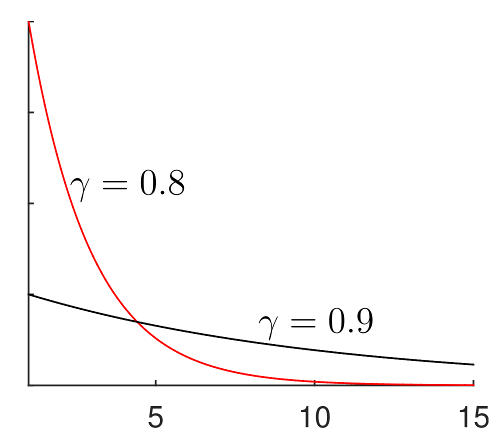
action $a_t \in A$

transition dynamics $P\,(\,\cdot \mid s_t\,,\,a_t)$

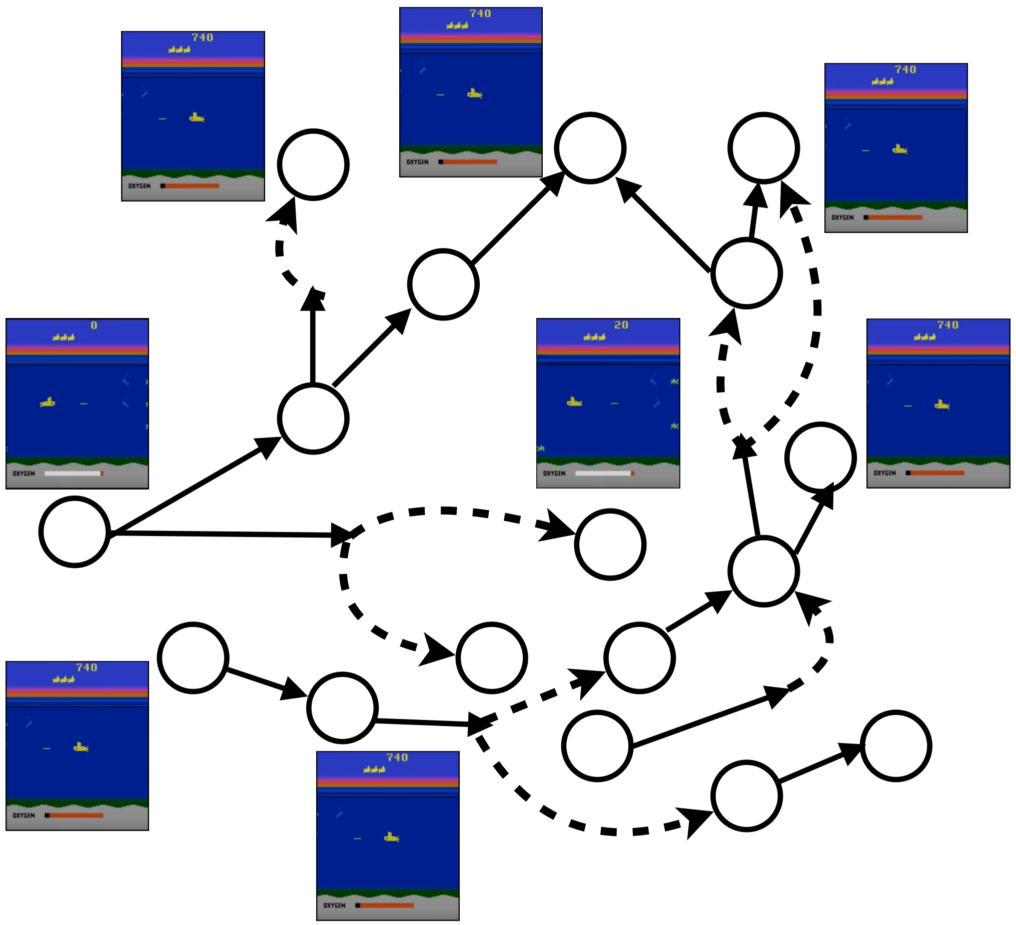
(unknown)

objective: maximize $\mathbb{E}\left[\sum_{t=1}^{H} r_t \mid \pi\right]$





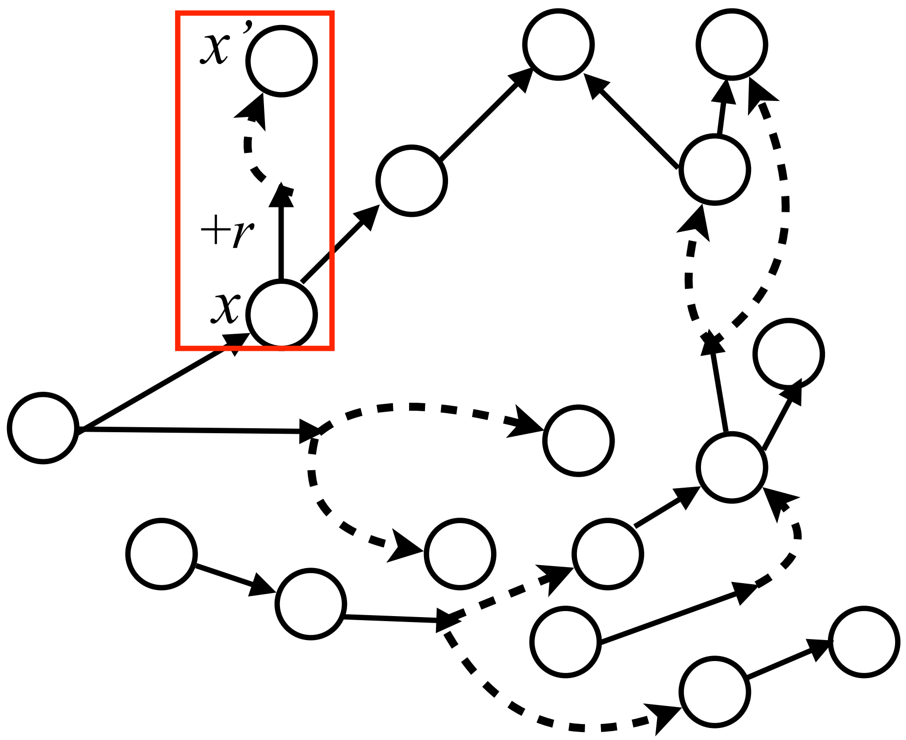
Video game playing

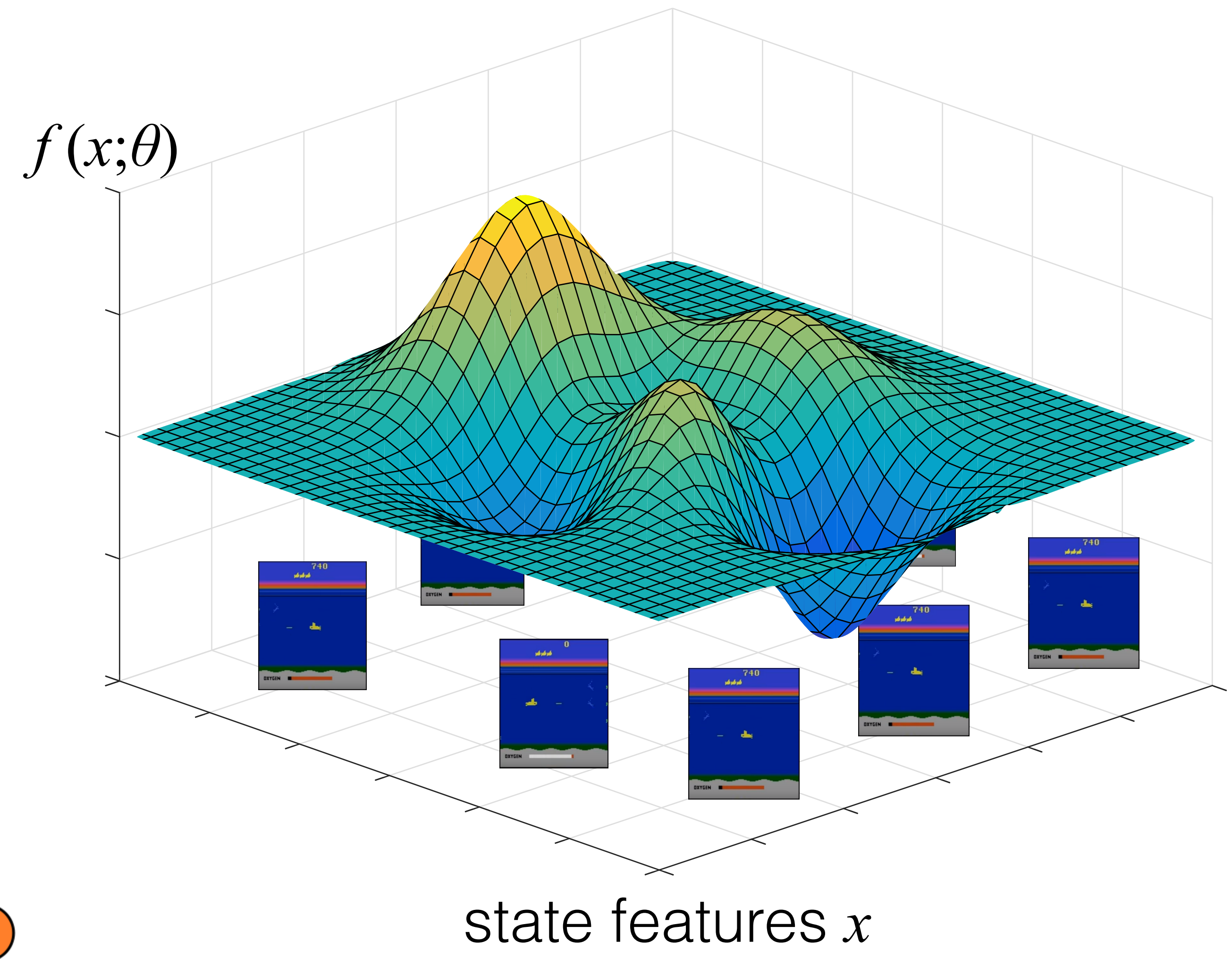


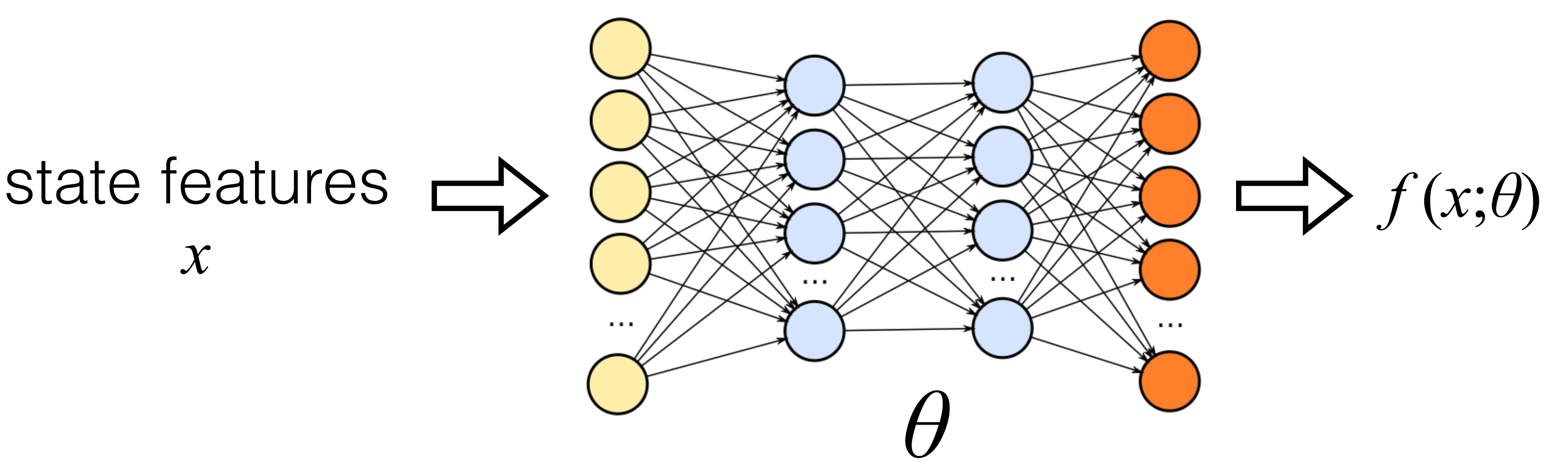
Need **generalization**

Value function approximation

Video game playing







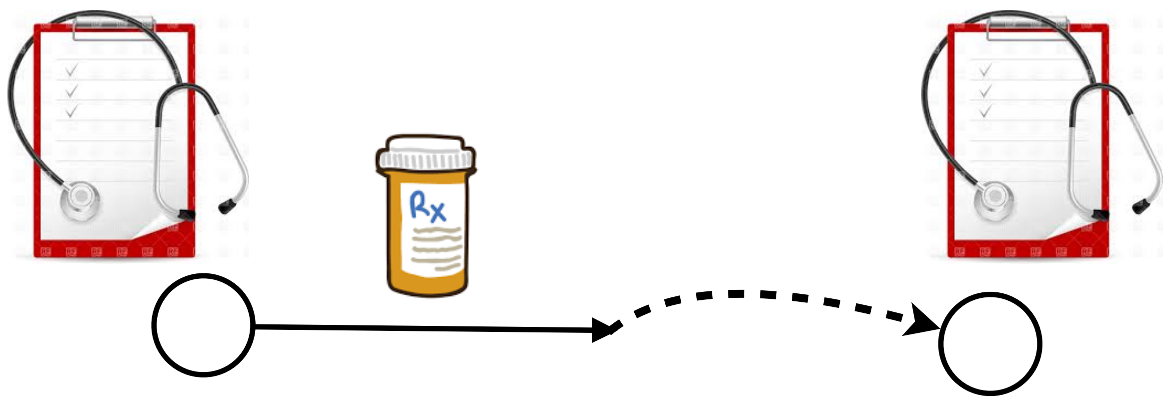
Find θ s.t.

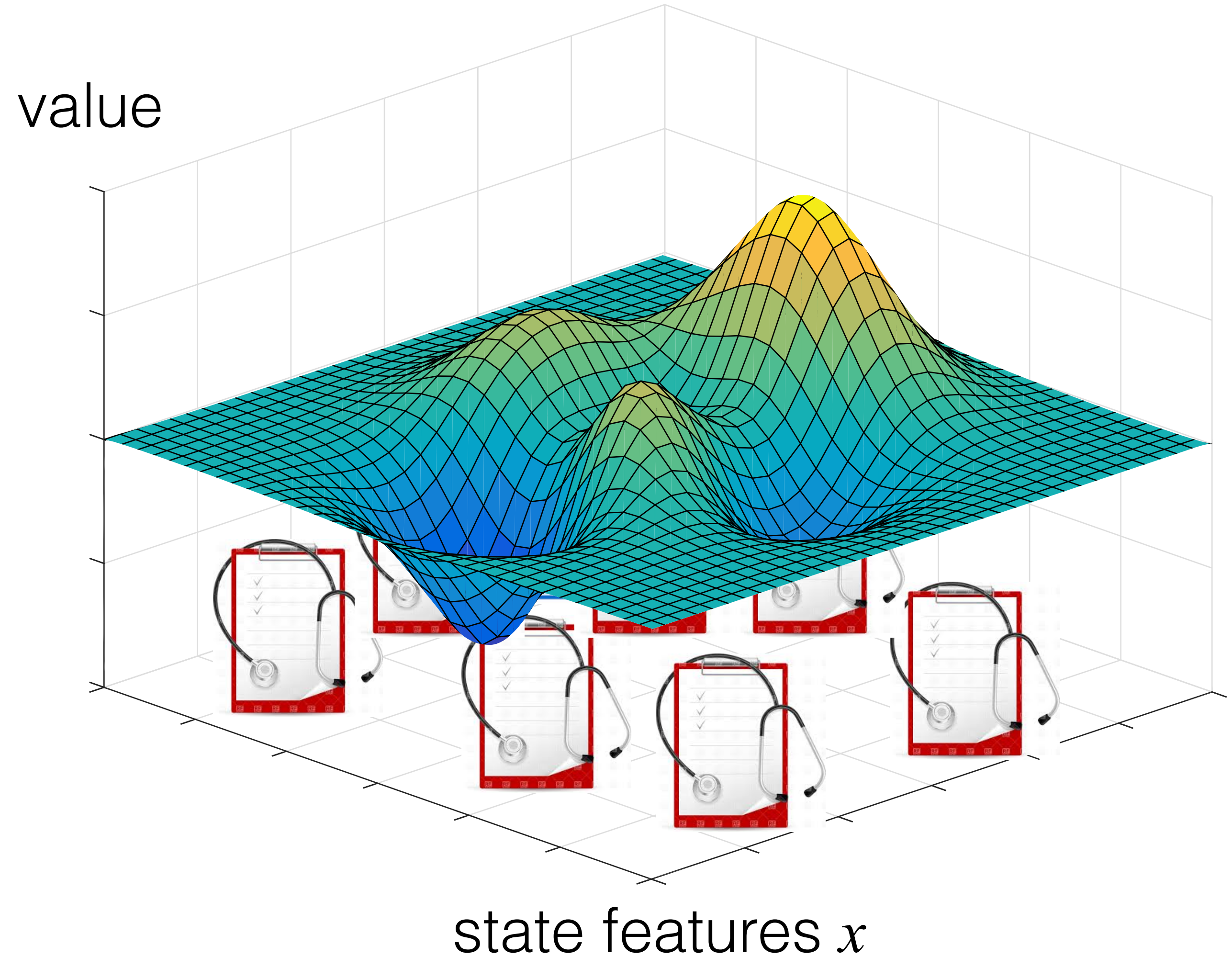
Need generalization

Value function approximation

$$f(\cdot\,;\theta) \approx V^*$$

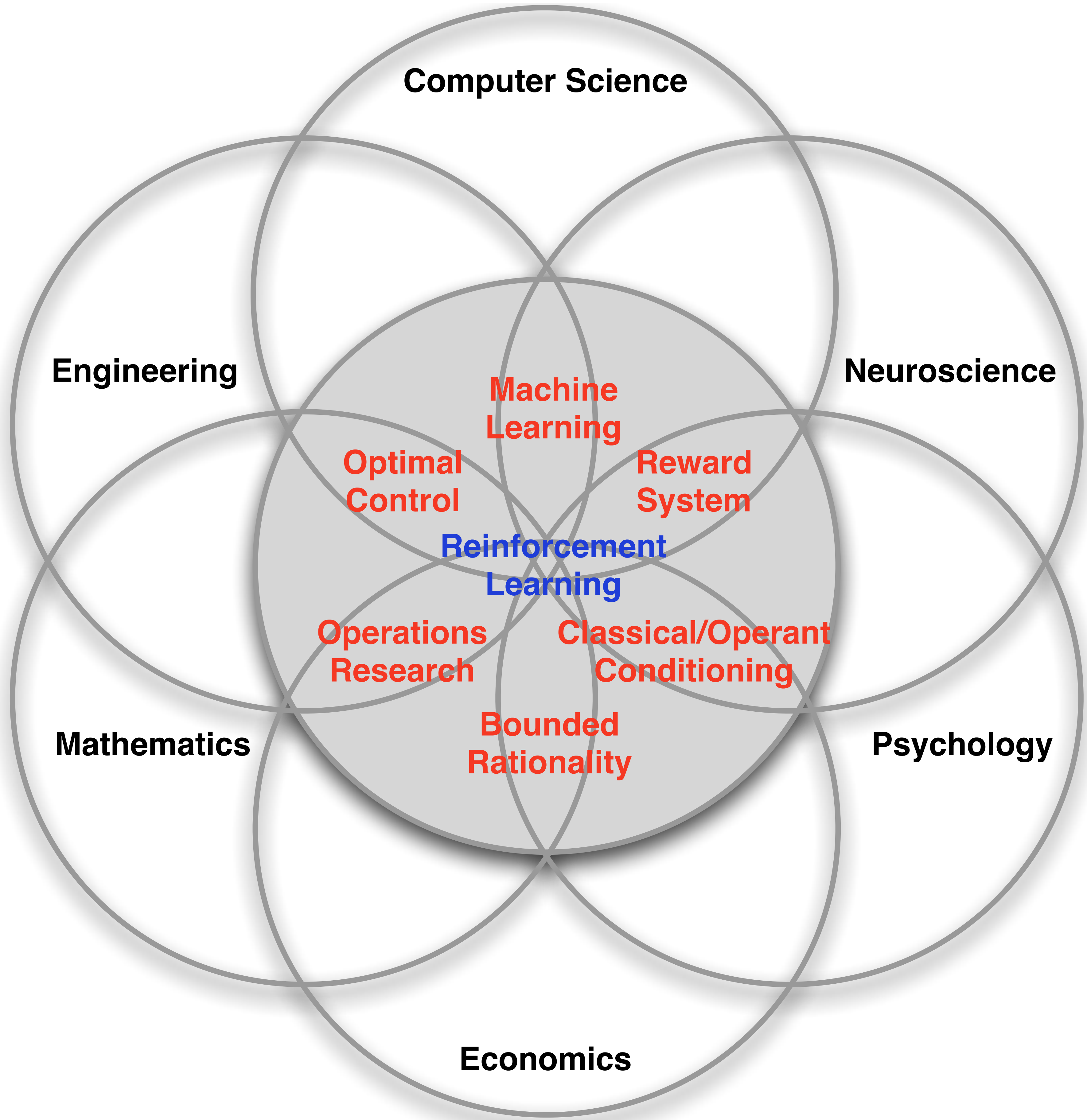
Adaptive medical treatment





- State: diagnosis
- Action: treatment
- Reward: progress in recovery

A Machine Learning view of RL



slide credit: David Silver

Supervised Learning

Given $\{(x^{(i)}, y^{(i)})\}$, learn $f : x \mapsto y$

- Online version: for round $t = 1, 2, \ldots,$ the learner
  - observes $x^{(t)}$
  - predicts $\hat{y}^{(t)}$
  - receives $y^{(t)}$
- Want to maximize # of correct predictions
- e.g., classifies if an image is about a dog, a cat, a plane, etc. (multi-class classification)
- Dataset is fixed for everyone
- “Full information setting”
- Core challenge: generalization

Contextual bandits

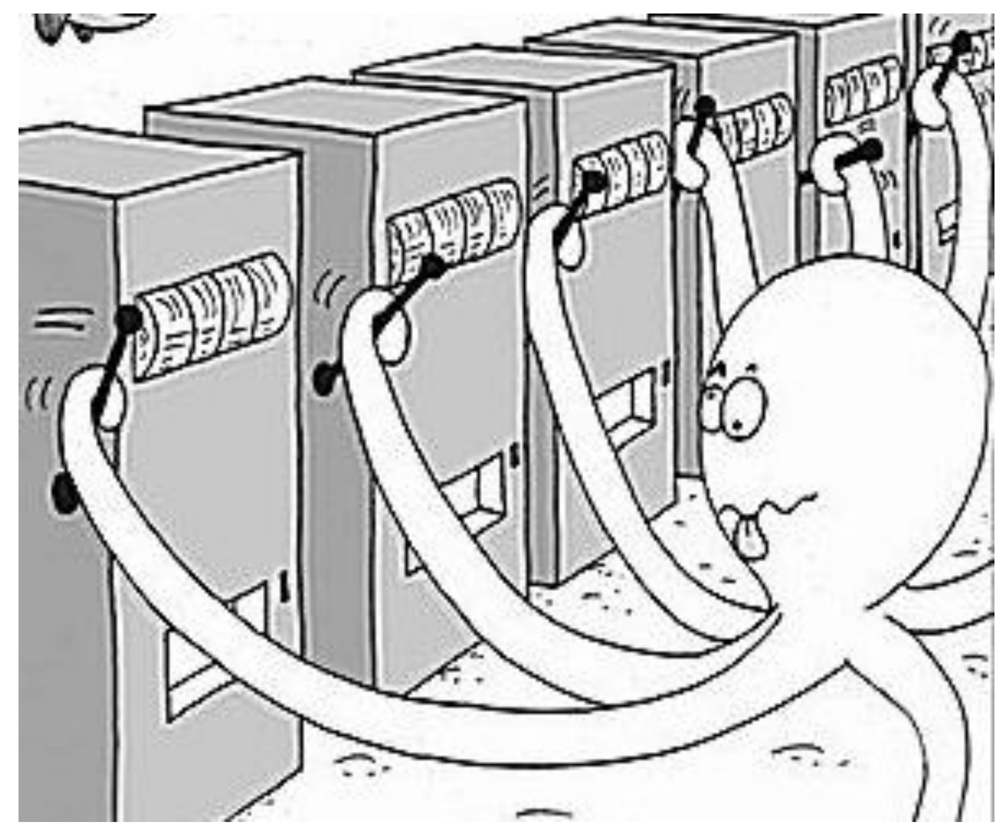
For round $t = 1, 2, \ldots,$ the learner

- Given $x^{(t)}$, chooses from a set of actions $a^{(t)} \in A$
- Receives reward $r^{(t)} \sim R(x^{(t)}, a^{(t)})$ (i.e., can be random)
- Want to maximize total reward
- You generate your own dataset $\{(x^{(t)}, a^{(t)}, r^{(t)})\}$!
- e.g., for an image, the learner guesses a label, and is told whether correct or not (reward = 1 if correct and 0 otherwise). Do not know what's the true label.
- e.g., for an user, the website recommends a movie, and observes whether the user likes it or not. Do not know what movies the user really want to see.
- "Partial information setting"

Contextual bandits

Contextual Bandits (cont.)

- Simplification: no x, Multi-Armed Bandits (MAB)
- Bandit is a research area by itself. we will not do a lot of bandits but may go through some material that have important implications on general RL (e.g., lower bounds)



RL

For round $t = 1, 2, \ldots,$

- For time step $h = 1, 2, \ldots, H$, the learner
  - Observes $x_h^{(t)}$
  - Chooses $a_h^{(t)}$
  - Receives $r_h^{(t)} \sim R(x_h^{(t)}, a_h^{(t)})$
  - Next $x_{h+1}^{(t)}$ is generated as a function of $x_h^{(t)}$ and $a_h^{(t)}$ (or sometimes, all previous x's and a's within round t)
- Bandits + "Delayed rewards/consequences"
- The protocol here is for episodic RL (each t is an *episode*).

Why statistical RL?

Two types of scenarios in RL research

1. Solving a large planning problem using a learning approach
   - e.g., AlphaGo, video game playing, simulated robotics
   - Transition dynamics (Go rules) known, but too many states
   - Run the simulator to collect data
2. Solving a learning problem
   - e.g., adaptive medical treatment
   - Transition dynamics unknown (and too many states)
   - Interact with the environment to collect data

Why statistical RL?

Two types of scenarios in RL research

1. Solving a large planning problem using a learning approach
2. Solving a learning problem

- #2 is less studied & many challenges. Data (real-world interactions) is highest priority. Computation second.
- Even for #1, sample complexity lower bounds computational complexity, so sample efficiency is also important.

MDP Planning

Infinite-horizon discounted MDPs

An MDP $M = (S, A, P, R, \gamma)$

- State space S.
- Action space A.

We will only consider discrete and finite spaces in this course.

- Transition function $P : S \times A \to \Delta(S)$. $\Delta(S)$ is the probability simplex over S, i.e., all non-negative vectors of length $|S|$ that sums up to 1
- Reward function R: $S \times A \to \mathbb{R}$. (deterministic reward function)
- Discount factor $\gamma \in [0,1)$
- The agent starts in some state s_1, takes action a_1, receives reward $r_1 \sim R(s_1, a_1)$, transitions to $s_2 \sim P(s_1, a_1)$, takes action a_2, so on so forth — the process continues indefinitely

Value and policy

- Want to take actions in a way that maximizes value (or return):

$$\mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t\right]$$

  - This value depends on where you start and how you act
- Often assume boundedness of rewards: $r_t \in [0, R_{\max}]$
  - What's the range of $\mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t\right]$? $\left[0, \frac{R_{\max}}{1-\gamma}\right]$
- A (deterministic) policy $\pi: S \rightarrow A$ describes how the agent acts: at state s_t, chooses action $a_t = \pi(s_t)$.
- More generally, the agent may choose actions randomly ($\pi: S \rightarrow \Delta(A)$), or even in a way that varies across time steps ("non-stationary policies")
- Define $V^{\pi}(s) = \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \,\middle|\, s_1 = s, \pi\right]$

Bellman equation for policy evaluation

$$
\begin{aligned}
V^\pi(s) &= \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \;\middle|\, s_1 = s, \pi\right] \\
&= \mathbb{E}\left[r_1 + \sum_{t=2}^{\infty} \gamma^{t-1} r_t \;\middle|\, s_1 = s, \pi\right] \\
&= R(s, \pi(s)) + \sum_{s' \in \mathcal{S}} P(s'|s, \pi(s))\, \mathbb{E}\left[\gamma \sum_{t=2}^{\infty} \gamma^{t-2} r_t \;\middle|\, s_1 = s, s_2 = s', \pi\right] \\
&= R(s, \pi(s)) + \sum_{s' \in \mathcal{S}} P(s'|s, \pi(s))\, \mathbb{E}\left[\gamma \sum_{t=2}^{\infty} \gamma^{t-2} r_t \;\middle|\, s_2 = s', \pi\right] \\
&= R(s, \pi(s)) + \gamma \sum_{s' \in \mathcal{S}} P(s'|s, \pi(s))\, \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \;\middle|\, s_1 = s', \pi\right] \\
&= R(s, \pi(s)) + \gamma \sum_{s' \in \mathcal{S}} P(s'|s, \pi(s))\, V^\pi(s') \\
&= R(s, \pi(s)) + \gamma \langle P(\cdot|s, \pi(s)),\; V^\pi(\cdot) \rangle
\end{aligned}
$$

Bellman equation for policy evaluation

$$V^\pi(s) = R(s, \pi(s)) + \gamma \langle P(\,\cdot \mid s, \pi(s)), V^\pi(\,\cdot\,) \rangle$$

Matrix form: define

- V^π as the $|S| \times 1$ vector $[V^\pi(s)]_{s \in S}$
- R^π as the vector $[R(s, \pi(s))]_{s \in S}$
- P^π as the matrix $[P(s' \mid s, \pi(s))]_{s \in S,\, s' \in S}$

$$V^\pi = R^\pi + \gamma P^\pi V^\pi$$

$$(I - \gamma P^\pi) V^\pi = R^\pi$$

$$V^\pi = (I - \gamma P^\pi)^{-1} R^\pi$$

This is always invertible. Proof?

Generalize to stochastic policies

- If $\pi : \mathcal{S} \to \Delta(\mathcal{A})$

$$V^\pi(s) = \mathbb{E}_{a \sim \pi(\cdot|s)}[R(s,a)] + \gamma \sum_{a \in \mathcal{A}, s' \in \mathcal{S}} \pi(a|s) P(s'|s,a) V^\pi(s')$$

$$= \mathbb{E}_{a \sim \pi(\cdot|s), s' \sim P(\cdot|s,a)}[R(s,a) + \gamma V^\pi(s')]$$

- Matrix form $V^\pi = R^\pi + \gamma P^\pi V^\pi$ still holds with

$$R^\pi(s) = \mathbb{E}_{a \sim \pi(\cdot|s)}[R(s,a)]$$

Shorthand: $R(s, \pi)$

$$P^\pi(s'|s) = \sum_{a \in \mathcal{A}} \pi(a|s) P(s'|s,a)$$

Shorthand: $P(s' \mid s, \pi)$

- Convention: “(s)” after π dropped & integration over action implicit

Homework 0

- uploaded on course website
- help understand the relationships between alternative MDP formulations
- more like readings w/ questions to think about
- no need to submit

State occupancy

$$(1-\gamma)\cdot(I-\gamma P^{\pi})^{-1}$$

Each row (indexed by s) is the normalized discounted state occupancy $d^{\pi,s}$, whose (s')-th entry is

$$d^{\pi,s}(s') = (1-\gamma)\cdot \sum_{t=1}^{\infty} \gamma^{t-1} d_t^{\pi,s}$$

$$\text{where} \quad d_t^{\pi,s}(s') = \mathbb{P}^{\pi}[s_t = s' \,\Big|\, s_1 = s\,]$$

- $V^{\pi}(s) = \sum_{t=1}^{\infty} \gamma^{t-1}\mathbb{E}\left[r_t \,\Big|\, s_1 = s, \pi\right] = \sum_{t=1}^{\infty} \gamma^{t-1} \sum_{s'} \mathbb{P}^{\pi}[s_t = s' \,|\, s_1 = s] R(s,\pi)$
- Also: $(I-\gamma P^{\pi})^{-1} = \sum_{t=1}^{\infty} \gamma^{t-1}(P^{\pi})^{t-1}$, and $(P^{\pi})^{t-1}(s' \,|\, s) = \mathbb{P}^{\pi}[s_t = s' \,|\, s_1 = s]$
- $(1\text{-}\gamma)$ is the normalization factor so that matrix is row-stochastic.
- Can also be interpreted as the value function of indicator reward

Optimality

- For infinite-horizon discounted MDPs, there always exists a stationary and deterministic policy that is optimal for all starting states simultaneously
  - Proof: Puterman'94, Thm 6.2.7 (reference due to Shipra Agrawal)
- Let π^* denote this optimal policy, and $V^* := V^{\pi^*}$
- Bellman Optimality Equation:

$$V^\star(s) = \max_{a \in A} \left(R(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \left[V^\star(s') \right] \right)$$

- If we know V^*, how to get π^* ?
- Easier to work with Q-values: $Q^*(s,a)$, as $\pi^\star(s) = \arg\max_{a \in A} Q^\star(s,a)$

$$Q^\star(s,a) = R(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \left[\max_{a' \in A} Q^\star(s',a') \right]$$