Concentration ineq. Hoeffding's Ineq

Theorem 1. *Let $X_1, \ldots, X_n$ be independent random variables on $\mathbb{R}$ such that X_i is bounded in the interval $[a_i, b_i]$. Let $S_n = \sum_{i=1}^n X_i$. Then for all $t > 0$,* bad event.

$$\underbrace{\Pr[S_n - \mathbb{E}[S_n] \geq t]} \leq \boxed{e^{-2t^2/\sum_{i=1}^n (b_i - a_i)^2}} \tag{1}$$

$$\Pr[S_n - \mathbb{E}[S_n] \leq -t] \leq e^{-2t^2/\sum_{i=1}^n (b_i - a_i)^2} \tag{2}$$

(1) $\Pr\left[(S_n - \mathbb{E}[S_n] \geq t) \cup (\cdots \leq -t) \right]$

$\leq \Pr[\cdots \geq \boxed{t}] + \Pr[\cdots \leq -t]$

$\leq \boxed{2 \cdot (\text{RHS of } (1))}$ δ.

$P(A \cup B) \leq P(A) + P(B)$.

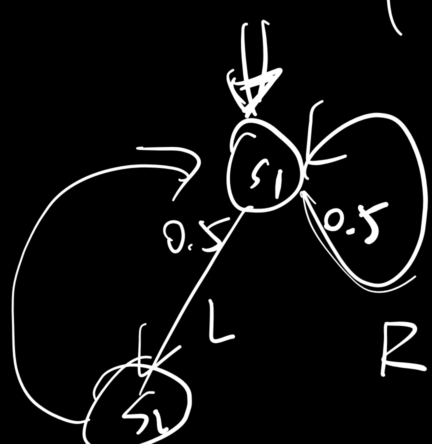
(2) Or, equivalently: failure prob.

fix any $0 < \delta < 1$. with prob. $\geq 1 - \delta$.

$$\left| \underbrace{\frac{S_n}{n} - \frac{\mathbb{E}[S_n]}{n}} \right| \leq \boxed{(b-a)} \sqrt{\frac{1}{2n} \ln \frac{2}{\delta}}.$$

(when $a_i \equiv a$, $b_i \equiv b$).

(3).



$\underbrace{s_1 \; s_1 \; s_2 \; s_1 \; s_1 \; s_2 \; s_1 \; s_2 \; s_3,}_{1000.}$

600.

$$1 \times \#R + 2 \times \#L = 1000.$$

$$\#R + \#L = 600$$

Scenario 2: run the chain until you hit s_1 600 times.

Application in bandits

For $t = 1, 2, 3, \ldots, T$

1. Learner chooses $i_t \in [K] := \{1, 2, \ldots, K\}$.

   action space A

2. Observes $r_t \sim R_{i_t}$

   r.v. bounded in $[0,1]$.

- Let μ_i denote mean of R_i.

Alg: (1) pick each arm T/K times.

(2) $\hat{\mu}_i$

(3) Output $\hat{i} = \arg\max_{i \in [K]} \hat{\mu}_i$.

Analysis: how good is $\hat{i}$? $\mu^* - \mu_{\hat{i}} \le ?$

$$\mu^* = \max_i \mu_i.$$

For any i, w.p. $\geq 1-\delta$, $|\hat{\mu}_i - \mu_i| \leq \sqrt{\frac{1}{2T/K} \ln \frac{2}{\delta}}$.

$\Downarrow$

w.p. $\geq 1-\delta$, $\forall i$, $|\hat{\mu}_i - \mu_i| \leq \sqrt{\frac{K}{2T} \ln \frac{2K}{\delta}}$.

?

$$\Pr\left[\boxed{\bigcup \left(|\hat{\mu}_i - \mu_i| > \text{blah}\right)}\right].$$

$$\leq \underbrace{\sum_{i=1}^{K} \Pr\left[|\hat{\mu}_i - \mu_i| > \sqrt{\frac{K}{2T} \ln \frac{2}{\delta'}}\right]}_{\delta' = \delta / K.} \leq K\delta' = \delta.$$

$$\Rightarrow \boxed{\text{w.p.} \geq 1-\delta, \ \forall i \ |\hat{\mu}_i - \mu_i| \leq \sqrt{\frac{K}{2T} \ln \frac{2K}{\delta}}.}$$

$$\mu^* - \mu_{\hat{i}} = \mu_{\underset{\substack{\uparrow \\ \arg\max_i \mu_i}}{i^*}} - \mu_{\hat{i}}$$

$$= \underbrace{\mu_{i^*} - \hat{\mu}_{i^*}} + \hat{\mu}_{i^*} - \mu_{\hat{i}}.$$

$$\leq \max_i |\mu_i - \hat{\mu}_i| + \hat{\mu}_{\hat{i}} - \mu_{\hat{i}}.$$

$$\overset{(1)}{\max_i} |\mu_i - \hat{\mu}_i|.$$

$$\leq 2 \cdot \sqrt{\frac{K}{2T} \ln \frac{2K}{\delta}} \cdot \Delta$$

Learning thry "101". / Assume $\mathcal{F}$ is finite.

Given iid (X_i, Y_i) pairs. $Y_i \in \{0,1\}$.

$$\mathcal{F} = \{ f : X \to \{0,1\} \}.$$

$$\hat{f} = \underset{f \in \mathcal{F}}{\operatorname{argmin}} \; \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}[f(X_i) \neq Y_i].$$

$$\boxed{\mathbb{P}[\hat{f}(X_i) \neq Y_i] - \mathbb{P}[f^*(X_i) \neq Y_i].}$$

where $f^* = \underset{f \in \mathcal{F}}{\operatorname{argmin}} \; \mathbb{P}[f(X_i) \neq Y_i].$

Fix $f \in \mathcal{F}$. w.p. $\geq 1-\delta$.

$\hat{\mathbb{P}}[f \dots]$

$$\left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}[f(X_i) \neq Y_i] - \mathbb{P}[f(X_i) \neq Y_i] \right| \leq \sqrt{\frac{1}{2n} \ln \frac{2}{\delta}}.$$

$$\Downarrow$$

w.p. $\geq 1-\delta$, $\forall f \in \mathcal{F}$. $\left| \cdots \right| \leq \sqrt{\frac{1}{2n} \ln \frac{2|\mathcal{F}|}{\delta}}$.

$$\mathbb{P}[\hat{f}(X_i) \neq Y_i] - \mathbb{P}[f^*(X_i) \neq Y_i]$$

$$= \underbrace{\mathbb{P}[\hat{f} \ldots] - \hat{\mathbb{P}}[\hat{f} \ldots]}_{\leq} + \underbrace{\hat{\mathbb{P}}[\hat{f} \ldots]}_{\leq} - \mathbb{P}[f^* \ldots]$$

$$\hat{\mathbb{P}}[\hat{f} \ldots] \to \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\hat{f}(X_i) \neq Y_i].$$

$$\leq \sqrt{\frac{1}{2n} \ln \frac{2|F|}{\delta}} + \underbrace{\hat{\mathbb{P}}[f^* \ldots] - \mathbb{P}[f^* \ldots]}$$

$$\leq \sqrt{\frac{2}{n} \ln \frac{2|F|}{\delta}}.$$

$$f_\theta.$$

$$\phi(x)^\top \theta \geq 0. \to 1$$

$$< 0 \to 0.$$

$$\left(\frac{1}{\varepsilon}\right)^d.$$