Notes on Fitted Q-iteration

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1 Analysis of FQI

Let $M = (\mathcal{S}, \mathcal{A}, P, R, \gamma, d_0)$ be an MDP, where d_0 is the initial distribution over states. Given a dataset $\{(s, a, r, s')\}$ generated from M and a Q-function class $\mathcal{F} \subset \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$, we want to analyze the guarantee of Fitted Q-Iteration. This note is inspired by and scrutinizes the results in Approximate Value/Policy Iteration literature [e.g., 1, 2, 3] under simplification assumptions.

Setup and Assumptions

1. $\mathcal{F}$ is finite but can be exponentially large.
2. Realizability: $Q^\star \in \mathcal{F}$.
3. Bellman completeness: $\forall f \in \mathcal{F}, \mathcal{T} f \in \mathcal{F}$. (For finite $\mathcal{F}$, this implies realizability.)
4. The dataset $D = \{(s, a, r, s')\}$ is generated as follows: $(s, a) \sim \mu$, $r \sim R(s, a)$, $s' \sim P(s, a)$. Define the empirical update $\widehat{\mathcal{T}}_{\mathcal{F}} f'$ as

$$\mathcal{L}_D(f; f') := \frac{1}{|D|} \sum_{(s,a,r,s')\in D} \left(f(s, a) - r - \gamma V_{f'}(s')\right)^2 .$$
$$\widehat{\mathcal{T}}_{\mathcal{F}} f' := \arg\min_{f\in\mathcal{F}} \mathcal{L}_D(f; f'),$$

   where $V_{f'}(s') := \max_{a'} f'(s', a')$. Note that by completeness, $\mathcal{T} f' \in \mathcal{F}$ is the Bayes optimal regressor for the regression problem defined in $\mathcal{L}_D(f; f')$. It will also be useful to define

$$\mathcal{L}_\mu(f; f') := \mathbb{E}_D[\mathcal{L}_D(f; f')].$$

5. For any function $g : \mathcal{X} \to \mathbb{R}$, any distribution $\nu \in \Delta(\mathcal{X})$, and $p \geq 1$, define $\|g\|_{p,\nu} := (\mathbb{E}_{x\sim\nu}[|g(x)|^p])^{1/p}$, and let $\|g\|_\nu$ be a shorthand for $\|g\|_{2,\nu}$. Such norms are similarly defined for functions over $\mathcal{X}$.
6. Let d_h^π be the distribution of (s_h, a_h) under π, that is, $d_h^\pi(s, a) := \Pr[s_h = s, a_h = a \mid s_1 \sim d_0, \pi]$. d^π is the usual discounted occupancy. The same notations are sometimes abused to refer to the corresponding state marginals, which will be clarified if not clear from the context.

7. We call any state-action distribution **admissible** if it can be generated at some time step from d_0 in the MDP. That is, it takes the form of d_h^π for some h and (possibly non-stationary) policy π. Then, assume that data is exploratory: for any admissible ν,

$$\forall s \in \mathcal{S},\ \frac{\nu(s,a)}{\mu(s,a)} \leq C.$$

As a consequence, $\|\cdot\|_\nu \leq \sqrt{C}\|\cdot\|_\mu$. See slides for example scenarios where C is naturally bounded.

8. Algorithm (simplified for analysis): let $f_0 \equiv \mathbf{0}$ (assuming $\mathbf{0} \in \mathcal{F}$), and for $k \geq 1$, $f_k := \widehat{\mathcal{T}}_{\mathcal{F}} f_{k-1}$.

9. Uniform deviation bound (can be obtained by concentration inequalities and union bound):

$$\forall f, f' \in \mathcal{F}, |\mathcal{L}_D(f; f') - \mathcal{L}_\mu(f; f')| \leq \epsilon.$$

(Note: at the end we will show how to obtain fast rates.)

Goal Let $\hat{\pi} := \pi_{f_k}$. Derive an upper bound on $J(\pi^\star) - J(\hat{\pi})$.

Analysis

$$\begin{aligned}
J(\pi^\star) - J(\hat{\pi}) &= \sum_{h=1}^{\infty} \gamma^{h-1} \mathbb{E}_{s \sim d_h^{\hat{\pi}}}[V^\star(s) - Q^\star(s, \hat{\pi})] \\
&\leq \sum_{h=1}^{\infty} \gamma^{h-1} \mathbb{E}_{s \sim d_h^{\hat{\pi}}}[Q^\star(s, \pi^\star) - f_k(s, \pi^\star) + f_k(s, \hat{\pi}) - Q^\star(s, \hat{\pi})] \\
&\leq \sum_{h=1}^{\infty} \gamma^{h-1} \left(\|Q^\star - f_k\|_{1, d_h^{\hat{\pi}} \times \pi^\star} + \|Q^\star - f_k\|_{1, d_h^{\hat{\pi}} \times \hat{\pi}} \right) \\
&\leq \sum_{h=1}^{\infty} \gamma^{h-1} \left(\|Q^\star - f_k\|_{d_h^{\hat{\pi}} \times \pi^\star} + \|Q^\star - f_k\|_{d_h^{\hat{\pi}} \times \hat{\pi}} \right). \qquad (1)
\end{aligned}$$

In the above equations all the terms in the form of d_h^π should be treated as state distributions, and $d_h^\pi \times \pi'$ refers to a state-action distribution generated as $s \sim d_h^\pi, a \sim \pi'(\cdot|s)$. The last line contains two terms, both in the form of $\|Q^\star - f_k\|_\nu$ with some admissible $\nu \in \Delta(\mathcal{S} \times \mathcal{A})$. So it remains to bound $\|Q^\star - f_k\|_\nu$ for any $\nu \in \Delta(\mathcal{S} \times \mathcal{A})$ that satisfies bullet 7.

First a helper lemma:

Lemma 1. *Define* $\pi_{f,f_k}(s) := \arg\max_{a \in \mathcal{A}} \max\{f(s,a), f_k(s,a)\}$. *Then we have* $\forall \tilde{\nu} \in \Delta(\mathcal{S})$,

$$\|V_f - V_{f_k}\|_{\tilde{\nu}} \leq \|f - f_k\|_{\tilde{\nu} \times \pi_{f,f_k}}.$$

Proof.

$$\begin{aligned}
\|V_f - V_{f_k}\|_{\tilde{\nu}}^2 &= \sum_{s \in \mathcal{S}} \tilde{\nu}(s) (\max_{a \in \mathcal{A}} f(s,a) - \max_{a' \in \mathcal{A}} f_k(s,a'))^2 \\
&\leq \sum_{s \in \mathcal{S}} \tilde{\nu}(s) (f(s, \pi_{f,f_k}) - f_k(s, \pi_{f,f_k}))^2 = \|f - f_k\|_{\tilde{\nu} \times \pi_{f,f_k}}^2. \qquad \square
\end{aligned}$$

Now we can bound $\|Q^\star - f_k\|_\nu$ using Lemma 1. Define $P(\nu)$ as a distribution over $\mathcal{S}$ generated as $s' \sim P(\nu) \Leftrightarrow (s,a) \sim \nu, s' \sim P(s,a)$, and

$$\begin{aligned}
\|f_k - Q^\star\|_\nu &= \|f_k - \mathcal{T} f_{k-1} + \mathcal{T} f_{k-1} - Q^\star\|_\nu \\
&\le \|f_k - \mathcal{T} f_{k-1}\|_\nu + \|\mathcal{T} f_{k-1} - \mathcal{T} Q^\star\|_\nu \\
&\le \sqrt{C}\,\|f_k - \mathcal{T} f_{k-1}\|_\mu + \gamma \|V_{f_{k-1}} - V^\star\|_{P(\nu)} && (*) \\
&\le \sqrt{C}\,\|f_k - \mathcal{T} f_{k-1}\|_\mu + \gamma \|f_{k-1} - Q^\star\|_{P(\nu)\times \pi_{f_{k-1},Q^\star}}. && \text{(Lemma 1)}
\end{aligned}$$

Step (*) holds because:

$$\begin{aligned}
\|\mathcal{T} f_{k-1} - \mathcal{T} Q^\star\|_\nu^2 &= \mathbb{E}_{(s,a)\sim\nu}\left[\left((\mathcal{T} f_{k-1})(s,a) - (\mathcal{T} Q^\star)(s,a)\right)^2\right] \\
&= \mathbb{E}_{(s,a)\sim\nu}\left[\left(\gamma \mathbb{E}_{s'\sim P(s,a)}[V_{f_{k-1}}(s') - V^\star(s')]\right)^2\right] \\
&\le \gamma^2\, \mathbb{E}_{(s,a)\sim\nu, s'\sim P(s,a)}\left[\left(V_{f_{k-1}}(s') - V^\star(s')\right)^2\right] && \text{(Jensen)} \\
&= \gamma^2\, \mathbb{E}_{s'\sim P(\nu)}\left[\left(V_{f_{k-1}}(s') - V^\star(s')\right)^2\right] = \gamma^2\, \|V_{f_{k-1}} - V^\star\|_{P(\nu)}^2.
\end{aligned}$$

Note that we can apply the same analysis on $P(\nu)\times\pi_{f_{k-1},Q^\star}$ since it is also admissible, and expand the inequality k times. It then suffices to upper bound $\|f_k - \mathcal{T} f_{k-1}\|_\mu$.

$$\begin{aligned}
\|f_k - \mathcal{T} f_{k-1}\|_\mu^2 &= \mathcal{L}_\mu(f_k; f_{k-1}) - \mathcal{L}_\mu(\mathcal{T} f_{k-1}; f_{k-1}) && (\mathcal{L} \text{ squared loss + } \mathcal{T} f_{k-1} \text{ Bayes optimal}) \\
&\le \mathcal{L}_D(f_k; f_{k-1}) - \mathcal{L}_D(\mathcal{T} f_{k-1}; f_{k-1}) + 2\epsilon && (\mathcal{T} f_{k-1} \in \mathcal{F}) \\
&\le 2\epsilon. && (f_k \text{ minimizes } \mathcal{L}_D(\cdot\,; f_{k-1}))
\end{aligned}$$

Note that the RHS does not depend on k, so we conclude that for any admissible ν,

$$\|f_k - Q^\star\|_\nu \le \frac{1-\gamma^k}{1-\gamma}\sqrt{2C\epsilon} + \gamma^k V_{\max}.$$

Apply this to Equation (1) and we get

$$J(\pi^\star) - J(\pi_{f_k}) \le \frac{2}{1-\gamma}\left(\frac{1-\gamma^k}{1-\gamma}\sqrt{2C\epsilon} + \gamma^k V_{\max}\right).$$

Extension: fast rate The previous bound should have $O(n^{-1/4})$ dependence on sample size $n := |D|$, because ϵ in bullet 9 should be $O(n^{-1/2})$ using Hoeffding's, and the final bound depends on $\sqrt{\epsilon}$. Here we exploit realizability to achieve fast rate so that the final bound is $O(n^{-1/2})$.

Define

$$Y(f; f') := (f(s,a) - r - \gamma V_{f'}(s'))^2 - ((\mathcal{T} f')(s,a) - r - \gamma V_{f'}(s'))^2.$$

Plug each $(s,a,r,s') \in D$ into $Y(f;f')$ and we get i.i.d. variables $Y_1(f;f'), Y_2(f;f'), \ldots, Y_n(f;f')$ where $n = |D|$. It is easy to see that

$$\frac{1}{n}\sum_{i=1}^{n} Y_i(f;f') = \mathcal{L}_D(f;f') - \mathcal{L}_D(\mathcal{T} f'; f'),$$

so we only shift our objective $\mathcal{L}_D$ by a f-independent constant. Our goal is to show that

$$\|\widehat{\mathcal{T}}_{\mathcal{F}} f' - \mathcal{T} f'\|_\mu^2 \equiv \mathbb{E}[Y(\widehat{\mathcal{T}}_{\mathcal{F}} f'; f')] = O(1/n).$$

Note that this result can be directly plugged into the previous analysis by letting $f' = f_{k-1}$ (hence $\widehat{\mathcal{T}}_{\mathcal{F}} f' = f_k$), and we immediately obtain a final bound of $O(n^{-1/2})$.

To prove the result, first notice that $\forall f \in \mathcal{F}$,

$$\mathbb{E}[Y(f; f')] = \mathcal{L}_\mu(f; f') - \mathcal{L}_\mu(\mathcal{T} f'; f') = \|f - \mathcal{T} f'\|_\mu^2,$$

thanks to realizability and squared loss. Next we bound variance of Y:

$$\begin{aligned}
\mathbb{V}[Y(f; f')] &\leq \mathbb{E}[Y(f; f')^2] \\
&= \mathbb{E}\left[\left(\left(f(s,a) - r - \gamma V_{f'}(s')\right)^2 - \left((\mathcal{T} f')(s,a) - r - \gamma V_{f'}(s')\right)^2\right)^2\right] \\
&= \mathbb{E}\left[\left(f(s,a) - (\mathcal{T} f')(s,a)\right)^2 \left(f(s,a) + (\mathcal{T} f')(s,a) - 2r - 2\gamma V_{f'}(s')\right)^2\right] \\
&\leq 4V_{\max}^2 \, \mathbb{E}\left[\left(f(s,a) - (\mathcal{T} f')(s,a)\right)^2\right] \\
&= 4V_{\max}^2 \, \|f - \mathcal{T} f'\|_\mu^2 = 4V_{\max}^2 \, \mathbb{E}[Y(f; f')],
\end{aligned}$$

where $V_{\max} = R_{\max}/(1-\gamma)$ is a constant.

Next we apply (one-sided) Bernstein's inequality (see [4]) and union bound over all $f \in \mathcal{F}$. Let $N = |\mathcal{F}|$. For any fixed f', with probability at least $1 - \delta$, $\forall f \in \mathcal{F}$,

$$\begin{aligned}
\mathbb{E}[Y(f; f')] - \frac{1}{n}\sum_{i=1}^n Y_i(f; f') &\leq \sqrt{\frac{2\mathbb{V}[Y(f; f')] \log \frac{N}{\delta}}{n}} + \frac{4V_{\max}^2 \log \frac{N}{\delta}}{3n} && (Y_i \in [-V_{\max}^2, V_{\max}^2]) \\
&\leq \sqrt{\frac{8V_{\max}^2 \mathbb{E}[Y(f; f')] \log \frac{N}{\delta}}{n}} + \frac{4V_{\max}^2 \log \frac{N}{\delta}}{3n}.
\end{aligned}$$

Since $\widehat{\mathcal{T}}_{\mathcal{F}} f'$ minimizes $\mathcal{L}_D(\,\cdot\,; f')$, it also minimizes $\frac{1}{n}\sum_{i=1}^n Y_i(\cdot; f')$ because the two objectives only differ by a constant $\mathcal{L}_D(\mathcal{T} f'; f')$. Hence,

$$\frac{1}{n}\sum_{i=1}^n Y_i(\widehat{\mathcal{T}}_{\mathcal{F}} f'; f') \leq \frac{1}{n}\sum_{i=1}^n Y_i(\mathcal{T} f'; f') = 0.$$

Then,

$$\mathbb{E}[Y(\widehat{\mathcal{T}}_{\mathcal{F}} f'; f')] \leq \sqrt{\frac{8V_{\max}^2 \mathbb{E}[Y(\widehat{\mathcal{T}}_{\mathcal{F}} f'; f')] \log \frac{N}{\delta}}{n}} + \frac{4V_{\max}^2 \log \frac{N}{\delta}}{3n}.$$

Solving for the quadratic formula,

$$\mathbb{E}[Y(\widehat{\mathcal{T}}_{\mathcal{F}} f'; f')] \leq \left(\sqrt{2} + \sqrt{\tfrac{10}{3}}\right)^2 \frac{V_{\max}^2 \log \frac{N}{\delta}}{n}.$$

Relaxing the definition of C The assumption $\|\nu/\mu\|_\infty \leq C$ for all admissible ν can be relaxed. In particular, note that we always use this assumption in the form of

$$\|f - \mathcal{T} f'\|_\nu \leq \sqrt{C}\|f - \mathcal{T} f'\|_\mu$$

for some $f, f' \in \mathcal{F}$. We can therefore literally redefine C as an upper bound of

$$\max_{f, f' \in \mathcal{F}} \frac{\|f - \mathcal{T} f'\|_\nu^2}{\|f - \mathcal{T} f'\|_\mu^2}$$

for all admissible ν. Despite the straightforward relaxation, when $\mathcal{F}$ has some nice structural properties, this new definition can be significantly tighter than the old definition based on raw density ratios. For example, when $\mathcal{F}$ is induced from a bisimulation state abstraction (which satisfies completeness), the new definition measures density ratio between the distributions over *abstract* state-action pairs, which can be much smaller than that between the raw state-action pairs. More generally, $\mathcal{F}$ is linear and Bellman completeness is satisfied, $f - \mathcal{T} f'$ is also a linear function, and the new definition measures coverage in the linear feature space. See further discussion on this in Akshay's note.

2 Alternative Analysis

Below we sketch an alternative proof to the FQI guarantee. There are two motivations:

Error propagation along "simple" policies The error propagation in the above analysis of FQI was along a somewhat "ugly" set of policies in the form of $\pi_{f_k,Q^\star}$, which in each state takes the action that "witnesses" the inequality $|\max_a f(s,a) - \max_a f'(s,a)| \le \max_a |f(s,a) - f'(s,a)|$ for $f = f_k$ and $f' = Q^\star$. However, the error propagation in the ADP literature (e.g., [3]) only involved "simple" policies, such as π_f for some $f \in \mathcal{F}$ (and the concatenation of such policies at different time steps to form a non-stationary policy).

"Modern" error-propagation analysis Error propagation in RL theory were often done by recursive expansion in the "old" literature, and the above analysis also follows this style. However, we have also seen alternative proofs based on cleaner and more elegant tools. For example, it is easy to analyze the error propagation of the "minimax algorithm" $\arg\min_{f\in\mathcal{F}} \max_{g\in\mathcal{F}} \mathcal{L}_D(f;f) - \mathcal{L}_D(g;f)$ [5, 6] using the following lemma: $\forall \pi, f$

$$J(\pi) - J(\pi_f) \le \frac{1}{1-\gamma}(\mathbb{E}_{d^\pi}[\mathcal{T}f - f] + \mathbb{E}_{d^{\pi_f}}[f - \mathcal{T}f]). \tag{2}$$

Using this lemma is also well aligned with the first motivation, as it often produces simple policies on the RHS (which the data distribution needs to cover).

2.1 Performance guarantee for non-stationary FQI

A major difficulty in applying Eq.(2) to FQI is that it requires control over the Bellman error $\|f - \mathcal{T}f\|$, i.e., the learned function should be "self-consistent". However, in FQI, we only have control over $\|f_t - \mathcal{T}f_{t-1}\|$, i.e., the output function f_k is not necessarily consistent with itself, but consistent with its previous iterate f_{k-1}, which is further consistent with *its* previous iterate f_{k-2}, and so on.

To overcome this difficulty, we first consider a different output policy: $\pi_{f_{k:0}} := \pi_{f_k} \circ \pi_{f_{k-1}} \circ \cdots \circ \pi_{f_0}$. This is a non-stationary policy, and after π_{f_0} we take arbitrary actions.[1] We call FQI with such a policy (instead of π_{f_k}) *non-stationary FQI*. Similar to the situation in value iteration (see note1), such a non-stationary policy actually has better guarantees than the usual FQI policy and saves a factor of horizon[2], and is also easier to analyze.

In particular, we now can use (a variant of) Eq.(2), because $\pi_{f_{k:0}}$ is greedy w.r.t. a self-consistent function, namely $f_{k:0} := f_k \circ f_{k-1} \circ \cdots \circ f_0$! The unusual aspect here is that we typically consider all value functions as only functions of states and actions, i.e., they are stationary. Here, the function $f_k \circ f_{k-1} \circ \cdots \circ f_0$ itself is a non-stationary object. Also, compared to Eq.(2) we will also need to include truncation error here. The lemma is given below, with proof left as a homework exercise:

[1] The result might be improved (though it is unclear if the improvement is significant) if we produce a periodic policy that simply repeats $\pi_{f_{k:0}}$ forever [7].

[2] Another way to save this factor of horizon is to run the minimax algorithm [6].

Lemma 2 (Non-stationary variant of Eq.(2)). *Given an arbitrary sequence of functions $f_0, \ldots, f_k \in \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$ and any (non-stationary) comparator policy π, let $\widehat{\pi} := \pi_{f_{k:0}}$ (followed by arbitrary actions after $k+1$ steps)*

$$J(\pi) - J(\widehat{\pi}) \le \sum_{t=1}^{k} \gamma^{t-1} \left(\mathbb{E}_{d_t^\pi}[\mathcal{T} f_{k-t} - f_{k-t+1}] + \mathbb{E}_{d_t^{\widehat{\pi}}}[f_{k-t+1} - \mathcal{T} f_{k-t}]\right) + \gamma^k V_{\max}. \tag{3}$$

According to the RHS of the bound, when we choose the optimal policy $\pi^\star$ as the comparator policy π, we need the data μ to cover the state distributions induced by two types of policies from d_0: $(\pi^\star)^t$, $\forall t \le k$, and $\pi^{f_{k:k'}}$, $\forall 0 \le k' \le k$. Caution: when we analyze the minimax algorithm using Eq.(2), we only need the data μ to cover the discounted occupancy as a whole, instead of covering the per-step distributions that contribute to the occupancy separately. Here we do not enjoy such a property, because our algorithm controls $\|f_t - \mathcal{T} f_{t-1}\|_{2,\mu}$ for each t separately, so change of measure must happen in each step instead of over the entire occupancy as a whole.

2.2 Performance guarantee for FQI

The previous section sketches an analysis of non-stationary FQI. To relate FQI to the analysis of its non-stationary variant, we use performance difference lemma

$$\begin{aligned} J(\pi^\star) - J(\pi_{f_k}) &= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi_{f_k}}}[V^\star(s) - Q^\star(s, \pi_{f_k})] \\ &\le \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi_{f_k}}}[V^\star(s) - V^{\pi_{f_{k:0}}}(s)]. \end{aligned}$$

Here the inequality is due to $Q^\star(s, \pi_{f_k}) \ge V^{\pi_{f_{k:0}}}(s)$, because $Q^\star(s, \pi_{f_k})$ is the expected return of starting in s, takes π_{f_k} immediately (which coincides with $V^{\pi_{f_{k:0}}}(s)$ in the first time step because π_{f_k} is $\pi_{f_{k:0}}$'s first-step policy), and acts optimally thereafter.

Now, the RHS looks like the performance guarantee of $\pi_{f_{k:0}}$, which we can directly apply the analysis in the previous section! We can also see that compared to the guarantee of non-stationary FQI, here we paying an extra $1/(1-\gamma)$ factor. The only unusual aspect is that here $d^{\pi_{f_k}}$ is treated as the initial distribution for the non-stationary FQI analysis, so finally, the distributions that need to be covered are those induced by the policies mentioned at the end of Section 2.1, but from $d_{\pi_{f_k}}$ as the initial distribution.

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