

Notes on Tabular Methods

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1 Overview of the methods

1.1 Tabular certainty-equivalence

Certainty-equivalence is a **model-based** RL algorithm, that is, it first estimates an MDP model from data, and then performs policy evaluation or optimization in the estimated model as if it were true. To specify the algorithm it suffices to specify the model estimation step.

Given a dataset D of trajectories, $D = \{(s_1, a_1, r_1, s_2, \dots, s_{H+1})\}$, we first convert it into a bag of $\{(s, a, r, s')\}$ tuples, where each trajectory is broken into H tuples: $(s_1, a_1, r_1, s_2), (s_2, a_2, r_2, s_3), \dots, (s_H, a_H, r_H, s_{H+1})$. For every $s \in \mathcal{S}, a \in \mathcal{A}$, define $D_{s,a}$ as the subset of tuples where the first element of the tuple is s and the second is a , and we write $(r, s') \in D_{s,a}$ since all tuples in $D_{s,a}$ share the same state-action pair. The tabular certainty-equivalence model uses the following estimation of the transition function \hat{P} : let $\mathbf{e}_{s'}$ be the unit vector whose s' -th entry is 1 and all other entries are 0,

$$\hat{P}(s, a) = \frac{1}{|D_{s,a}|} \sum_{(r,s') \in D_{s,a}} \mathbf{e}_{s'}. \quad (1)$$

Here $\mathbb{I}(\cdot)$ is the indicator function. In words, $\hat{P}(s'|s, a)$ is simply the empirical frequency of observing s' after taking a in state s . Similarly when reward function also needs to be learned, the estimate is

$$\hat{R}(s, a) = \frac{1}{|D_{s,a}|} \sum_{(r,s') \in D_{s,a}} r. \quad (2)$$

\hat{P} and \hat{R} are the maximum likelihood estimates of the transition and the reward functions, respectively. Note that for the transition function to be well-defined we need $n(s, a) > 0$ for every $s, a \in \mathcal{S}$.

1.2 Value-based tabular methods

Certainty-equivalence explicitly stores an estimated MDP model, which has $O(|\mathcal{S}|^2|\mathcal{A}|)$ space complexity, and the algorithm has a *batch* nature, i.e., it is invoked after all the data are collected. In contrast, there is another popular family of RL algorithms that (1) only model the Q-value functions hence has $O(|\mathcal{S}||\mathcal{A}|)$ sample complexity, (2) can be applied in an online manner, i.e., the algorithm runs as more and more data are collected. Well-known examples include Q-learning [1] and Sarsa [2].

Another very appealing property of these methods is that it is relatively easy to incorporate sophisticated generalization schemes, such as deep neural networks, which has recently led to many

empirical successes [3]. On the other hand, such methods are typically less sample-efficient than model-based methods and will not be discussed in more details in this note.¹

2 Analysis of certainty-equivalence RL

Here we analyze the method introduced in Section 1.1. For simplicity we further assume that data are generated by sampling each (s, a) a fixed number of times. We are interested in deriving high-probability guarantees for the optimal policy of $\widehat{M} = (\mathcal{S}, \mathcal{A}, \widehat{P}, \widehat{R}, \gamma)$ as a function of $n \equiv |D_{s,a}|$.

We provide three different analyses for the algorithm, and we should see some interesting trade-off between state space and horizon.

2.1 Naive analysis

The basic idea is, when n is sufficiently large, we expect $\widehat{R} \approx R$ and $\widehat{P} \approx P$. In particular, by Hoeffding's inequality and union bound, the following inequalities hold with probability at least $1 - \delta$:

$$\max_{s,a} |\widehat{R}(s, a) - R(s, a)| \leq R_{\max} \sqrt{\frac{1}{2n} \ln \frac{4|\mathcal{S} \times \mathcal{A}|}{\delta}} \quad (3)$$

and

$$\max_{s,a,s'} |\widehat{P}(s'|s, a) - P(s'|s, a)| \leq \sqrt{\frac{1}{2n} \ln \frac{4|\mathcal{S} \times \mathcal{A} \times \mathcal{S}|}{\delta}}. \quad (4)$$

Note that we first split the failure probability δ evenly between the reward estimation events and the transition estimation events. Then for reward, we split $\delta/2$ evenly among all (s, a) ; for transition, we split $\delta/2$ evenly among all (s, a, s') . From Eq.(4) we further have

$$\max_{s,a} \|\widehat{P}(s, a) - P(s, a)\|_1 \leq \max_{s,a} |\mathcal{S}| \cdot \|\widehat{P}(s, a) - P(s, a)\|_\infty \leq |\mathcal{S}| \cdot \sqrt{\frac{1}{2n} \ln \frac{4|\mathcal{S} \times \mathcal{A} \times \mathcal{S}|}{\delta}}. \quad (5)$$

To bound the suboptimality of $\pi_{\widehat{M}}^*$, we first introduce the *simulation lemma* [6].

Lemma 1 (Simulation Lemma). *If $\max_{s,a} |\widehat{R}(s, a) - R(s, a)| \leq \epsilon_R$ and $\max_{s,a} \|\widehat{P}(s, a) - P(s, a)\|_1 \leq \epsilon_P$, then for any policy $\pi : \mathcal{S} \rightarrow \mathcal{A}$, we have*

$$\|V_{\widehat{M}}^\pi - V_M^\pi\|_\infty \leq \frac{\epsilon_R}{1-\gamma} + \frac{\gamma \epsilon_P V_{\max}}{2(1-\gamma)},$$

where $V_{\max} := R_{\max}/(1-\gamma)$.

¹Techniques such as experience replay can be used to improve the sample efficiency of many online algorithms [4], but it also blurs the boundary between value-based and model-based methods [5].

Proof. For any $s \in \mathcal{S}$,

$$\begin{aligned}
& |V_M^\pi(s) - V_M^\pi(s)| \\
&= |\widehat{R}(s, \pi) + \gamma \langle \widehat{P}(s, \pi), V_M^\pi \rangle - R(s, \pi) - \gamma \langle P(s, \pi), V_M^\pi \rangle| \\
&\leq \epsilon_R + \gamma |\langle \widehat{P}(s, \pi), V_M^\pi \rangle - \langle P(s, \pi), V_M^\pi \rangle + \langle P(s, \pi), V_M^\pi \rangle - \langle P(s, \pi), V_M^\pi \rangle| \\
&\leq \epsilon_R + \gamma |\langle \widehat{P}(s, \pi) - P(s, \pi), V_M^\pi \rangle| + \gamma \|V_M^\pi - V_M^\pi\|_\infty \\
&= \epsilon_R + \gamma |\langle \widehat{P}(s, \pi) - P(s, \pi), V_M^\pi - \frac{V_{\max}}{2} \cdot \mathbf{1} \rangle| + \gamma \|V_M^\pi - V_M^\pi\|_\infty \\
&\leq \epsilon_R + \gamma \|\widehat{P}(s, \pi) - P(s, \pi)\|_1 \|V_M^\pi - \frac{V_{\max}}{2} \cdot \mathbf{1}\|_\infty + \gamma \|V_M^\pi - V_M^\pi\|_\infty \\
&\leq \epsilon_R + \frac{\gamma \epsilon_P V_{\max}}{2} + \gamma \|V_M^\pi - V_M^\pi\|_\infty.
\end{aligned}$$

Since this holds for all $s \in \mathcal{S}$, we can also take infinite-norm on the LHS, which yields the desired result. Note that we subtract $\frac{V_{\max}}{2} \cdot \mathbf{1}$ ($\mathbf{1}$ is the all-one vector) to center the range of V_M^π around the origin, which exploits the fact that both $\widehat{P}(s, \pi)$ and $P(s, \pi)$ are valid probability distributions and sum up to 1. \square

The following lemma translates the policy evaluation error to the suboptimality of π_M^* :

Lemma 2 (Evaluation error to decision loss). $\forall s \in \mathcal{S}, V_M^*(s) - V_M^{\pi_M^*}(s) \leq 2 \sup_{\pi: \mathcal{S} \rightarrow \mathcal{A}} \|V_M^\pi - V_M^\pi\|_\infty$.

Proof. For any $s \in \mathcal{S}$,

$$\begin{aligned}
V_M^*(s) - V_M^{\pi_M^*}(s) &= V_M^{\pi_M^*}(s) - V_M^{\pi_M^*}(s) + V_M^{\pi_M^*}(s) - V_M^{\pi_M^*}(s) \\
&\leq V_M^{\pi_M^*}(s) - V_M^{\pi_M^*}(s) + V_M^{\pi_M^*}(s) - V_M^{\pi_M^*}(s) \quad (\pi_M^* \text{ maximizes } v_M) \\
&\leq \|V_M^{\pi_M^*} - V_M^{\pi_M^*}\|_\infty + \|V_M^{\pi_M^*} - V_M^{\pi_M^*}\|_\infty. \quad \square
\end{aligned}$$

Putting Lemmas 1 and 2 together with the concentration inequalities, we can see that the suboptimality we incur is

$$V_M^*(s) - V_M^{\pi_M^*}(s) = \tilde{O}\left(\frac{|\mathcal{S}|V_{\max}}{\sqrt{n}(1-\gamma)}\right), \forall s \in \mathcal{S}.$$

Here $\tilde{O}(\cdot)$ suppresses poly-logarithmic dependences on $|\mathcal{S}|$ and $|\mathcal{A}|$; in this note we also omit the dependence on R_{\max} and $1/\delta$, and only highlight the dependence on $|\mathcal{S}|$, n , and $1/(1-\gamma)$.

2.2 Improving $|\mathcal{S}|$ to $\sqrt{|\mathcal{S}|}$

The previous analysis proves concentration for each individual $P(s'|s, a)$ and adds up the errors to give an ℓ_1 error bound, which is loose. We can obtain a tighter analysis by proving an ℓ_1 concentration bound for multinomial distribution directly.

Note that for any vector $v \in \mathbb{R}^{|\mathcal{S}|}$,

$$\|v\|_1 = \sup_{u \in \{-1, 1\}^{|\mathcal{S}|}} u^\top v.$$

Each $u \in \{-1, 1\}^{|\mathcal{S}|}$ projects the vector v to some scalar value. If v can be written as the sum of zero-mean i.i.d. vectors, we can prove concentration for $u^\top v$ first, and then union bound over all u

to obtain the ℓ_1 error bound. Concretely, for any fixed (s, a) pair and any fixed $u \in \{-1, 1\}^{|\mathcal{S}|}$, with probability at least $1 - \delta/(2|\mathcal{S} \times \mathcal{A}| \cdot 2^{|\mathcal{S}|})$, we have

$$u^\top (\widehat{P}(s, a) - P(s, a)) \leq 2\sqrt{\frac{1}{2n} \ln \frac{2|\mathcal{S} \times \mathcal{A}| \cdot 2^{|\mathcal{S}|}}{\delta}}, \quad (6)$$

because $u^\top \widehat{P}(s, a)$ is the average of i.i.d. random variables $u^\top e_{s'}$ with bounded range $[-1, 1]$.² This leads to the following improvement over Eq.(5): w.p. at least $1 - \delta/2$,

$$\max_{s,a} \|\widehat{P}(s, a) - P(s, a)\|_1 = \max_{s,a} \max_{u \in \{-1, 1\}^{|\mathcal{S}|}} u^\top (\widehat{P}(s, a) - P(s, a)) \leq 2\sqrt{\frac{1}{2n} \ln \frac{2|\mathcal{S} \times \mathcal{A}| \cdot 2^{|\mathcal{S}|}}{\delta}}. \quad (7)$$

Roughly speaking, the $\tilde{O}(|\mathcal{S}| \sqrt{\frac{1}{n}})$ bound in Eq.5 is improved to $\tilde{O}(\sqrt{\frac{|\mathcal{S}|}{n}})$, and propagating the improvement through the remainder of the analysis yields

$$V_M^*(s) - V_M^{\pi_M^*}(s) = \tilde{O}\left(\frac{\sqrt{|\mathcal{S}|} V_{\max}}{\sqrt{n}(1-\gamma)}\right), \quad \forall s \in \mathcal{S}.$$

2.3 No dependence on $|\mathcal{S}|$

The last analysis removes the dependence of n on $|\mathcal{S}|$, at the cost of an additional dependence on $\frac{1}{1-\gamma}$. Note that the total number of samples still scales with $|\mathcal{S}|$ as we require n samples per (s, a) .

The core idea is to show $Q_M^* \approx Q_M^*$, and then upper bound loss by Lemma 4 from Note 1. First, by contraction we have,

$$\|Q_M^* - Q_M^*\|_\infty \leq \frac{1}{1-\gamma} \|Q_M^* - \mathcal{T}_{\widehat{M}} Q_M^*\|_\infty. \quad (8)$$

This is because

$$\begin{aligned} \|Q_M^* - Q_M^*\|_\infty &= \|\mathcal{T}_{\widehat{M}} Q_M^* - \mathcal{T}_{\widehat{M}} Q_M^* + \mathcal{T}_{\widehat{M}} Q_M^* - Q_M^*\|_\infty \\ &\leq \gamma \|Q_M^* - Q_M^*\|_\infty + \|\mathcal{T}_{\widehat{M}} Q_M^* - Q_M^*\|_\infty. \end{aligned} \quad (\mathcal{T}_{\widehat{M}} \text{ is a } \gamma\text{-contraction})$$

Then we bound the RHS in the following lemma.

Lemma 3. *For any fixed $s \in \mathcal{S}, a \in \mathcal{A}$, with probability at least $1 - \delta$,*

$$\left| Q_M^*(s, a) - \left(\widehat{R}(s, a) + \gamma \langle \widehat{P}(s, a), V_M^* \rangle \right) \right| \leq \frac{R_{\max}}{1-\gamma} \sqrt{\frac{1}{2n} \log \frac{2}{\delta}}.$$

Proof. The bound follows directly from Hoeffding's inequality upon the following observation:

$$\widehat{R}(s, a) + \gamma \langle \widehat{P}(s, a), V_M^* \rangle = \frac{1}{n} \sum_{(r, s') \in D_{s,a}} (r + \gamma V_M^*(s')).$$

Note that the RHS is the average of i.i.d. random variables $(r + \gamma V_M^*(s'))$ in the interval of $[0, \frac{R_{\max}}{1-\gamma}]$, whose expectation is exactly $Q_M^*(s, a)$. Therefore, the LHS of the lemma statement is the deviation of average of i.i.d. variables from the expectation, where Hoeffding's inequality applies. \square

²Also note that we only bound the deviation from one side, so we save a factor of 2 in \ln compared to bounding the absolute deviation. Another tiny improvement: for $u \in \{-1, 1\}^{|\mathcal{S}|}$, one can ignore $u = \pm \mathbf{1}$ as $\pm \mathbf{1}(\widehat{P}(s, a) - P(s, a)) \equiv 0$.

Note that the LHS of the lemma statement is simply the (s, a) -th entry of $(Q_M^* - \mathcal{T}_{\widehat{M}} Q_M^*)$. The final result we can get is

$$V_M^*(s) - V_M^{\pi_{\widehat{M}}}^*(s) = \tilde{O} \left(\frac{V_{\max}}{\sqrt{n}(1-\gamma)^2} \right), \forall s \in \mathcal{S}.$$

The cubic dependence on horizon comes from 3 different sources: (1) the range of value, (2) translating Bellman error to the difference in optimal Q-value functions, and (3) error accumulation over time when taking actions greedily wrt \widehat{Q} . The previous analyses only paid quadratic dependence on horizon because (3) was not present.

Some notes on Eq.(8) One can also obtain the following inequality by swapping the roles of M and \widehat{M} in Eq.(8):

$$\|Q_{\widehat{M}}^* - Q_M^*\|_\infty \leq \frac{1}{1-\gamma} \|Q_{\widehat{M}}^* - \mathcal{T}_M Q_{\widehat{M}}^*\|_\infty.$$

In fact, the RHS of the above inequality is the more standard notion of *Bellman errors* (or *Bellman residuals*): it measures how much an *approximate* Q-value function (here $Q_{\widehat{M}}^*$) deviates from itself when updated using the *true* Bellman update operator. In fact we can attempt to complete the analysis based on this inequality instead of Eq.(8), by noticing that the RHS is (ignoring $\frac{1}{1-\gamma}$ and the max-norm)

$$\mathcal{T}_{\widehat{M}} Q_{\widehat{M}}^* - \mathcal{T}_M Q_{\widehat{M}}^*.$$

This way we also introduce $\mathcal{T}_{\widehat{M}}$ into the expression and compare it with \mathcal{T}_M , which should allow us to use concentration inequalities to bound the difference between $\mathcal{T}_{\widehat{M}}$ and \mathcal{T}_M .

Now the (s, a) -th entry of the above expression is

$$\left(\widehat{R}(s, a) + \gamma \langle \widehat{P}(s, a), V_M^* \rangle \right) - \left(R(s, a) + \gamma \langle P(s, a), V_M^* \rangle \right)$$

It is attempting to use the techniques in the proof of Lemma 3, by claiming that $(r + V_M^*(s'))$ are i.i.d. random variables for $(r, s') \in D_{s,a}$, with expected value $R(s, a) + \gamma \langle P(s, a), V_M^* \rangle$. This is not true in general, because the function $V_M^*(s')$ itself is random and depends on the data in $D_{s,a}$! Hence Hoeffding does not apply. One workaround is to consider a deterministic function class that always contains V_M^* and do a union bound over that class; in fact, if we choose all tabular functions in the range of $[0, V_{\max}]$, the analysis is basically identical to Section 2.2.

Now you should see why we use Q_M^* and $\mathcal{T}_{\widehat{M}}$ in Eq.(8), as this way we compare \mathcal{T}_M and $\mathcal{T}_{\widehat{M}}$ against V_M^* , which is a *deterministic* function.

In cases where M 's state space forms a directed acyclic graph (DAG), the argument with V_M^* can still work as $V_M^*(s')$ only depends on the datasets for later state-action pairs, which do not include the current (s, a) under consideration. This argument is straightforward here because we have a very simple and clean data collection procedure. One has to be extremely careful when using this argument in more realistic settings: for example, in the exploration setting, even if $V_M^*(s')$ is estimated from datasets not including $D_{s,a}$, the outcomes in $D_{s,a}$ might have determined which later states we have sufficient samples and which not, which introduces very subtle interdependence with V_M^* .

Connection to MCTS Interestingly, the independence of n on $|\mathcal{S}|$ in the last analysis is the core idea that leads to Sparse Sampling [7], which is a prototype algorithm for the family of Monte-Carlo tree search algorithm that played a crucial role in the success of AlphaGo.

One way to view Sparse Sampling is the following: conceptually we run the tabular method with n set according to the last analysis (no dependence on $|\mathcal{S}|$). Of course, when $|\mathcal{S}|$ is large this is impractical, but if we only need to know $\pi^*(s_0)$ for some particular state s_0 (which is the setting of online planning with MCTS), we can perform “lazy evaluation”: only generate the datasets for state-action pairs that contribute to the calculation of $V_M^*(s_0)$ and truncate at the effective horizon. Roughly speaking, this requires a total of $(n|\mathcal{A}|)^{O(\frac{1}{1-\gamma})}$ samples to compute $\pi^*(s_0)$, where has no dependence on $|\mathcal{S}|$.

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