

Concentration Inequalities and Union Bound

Nan Jiang

September 13, 2021

This note introduces the basics of concentration inequalities and examples of its applications (often with union bound), which will be useful for the rest of this course.

1 Hoeffding's Inequality

Theorem 1. Let X_1, \dots, X_n be independent random variables on \mathbb{R} such that X_i is bounded in the interval $[a_i, b_i]$. Let $S_n = \sum_{i=1}^n X_i$. Then for all $t > 0$,

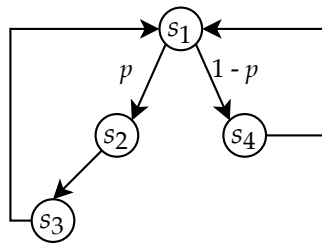
$$\Pr[S_n - \mathbb{E}[S_n] \geq t] \leq e^{-2t^2 / \sum_{i=1}^n (b_i - a_i)^2}, \quad (1)$$

$$\Pr[S_n - \mathbb{E}[S_n] \leq -t] \leq e^{-2t^2 / \sum_{i=1}^n (b_i - a_i)^2}. \quad (2)$$

Remarks:

- By union bound, we have $\Pr[|S_n - \mathbb{E}[S_n]| \geq t] \leq 2e^{-2t^2 / \sum_{i=1}^n (b_i - a_i)^2}$.
- We often care about the convergence of the empirical mean to the true average, so we can divide S_n by n : $\Pr\left[\left|\frac{S_n}{n} - \frac{\mathbb{E}[S_n]}{n}\right| \geq t\right] \leq 2e^{-2n^2 t^2 / \sum_{i=1}^n (b_i - a_i)^2}$.
- A useful rephrase of the result when all variables share the same support $[a, b]$: with probability at least $1 - \delta$, $\left|\frac{S_n}{n} - \frac{\mathbb{E}[S_n]}{n}\right| \leq (b - a)\sqrt{\frac{1}{2n} \ln \frac{2}{\delta}}$.
- X_1, \dots, X_n are not necessarily identically distributed; they just have to be independent.
- The number of variables, n , is a constant in the theorem statement. When n is a random variable itself, for Hoeffding's inequality to apply, n cannot depend on the realization of X_1, \dots, X_n .

Example: Consider the following Markov chain:



Say we start at s_1 and sample a path of length T (T is a constant). Let n be the number of times we visit s_1 , and we can use the transitions from s_1 to estimate p .

1. Can we directly apply Hoeffding's inequality here with n as the number of coin tosses? If you want to derive a concentration bound for this problem, look up Azuma's inequality.
2. What if we sample a path until we visit s_1 N times for some constant N ? Can we apply Hoeffding's inequality with N as the number of random variables?

2 Multi-Armed Bandits (MAB)

2.1 Formulation

A MAB problem is specified by K distributions over \mathbb{R} , $\{R_i\}_{i=1}^K$. Each R_i has bounded supported $[0, 1]$ and mean μ_i . Let $\mu^* = \max_{i \in [K]} \mu_i$. For round $t = 1, 2, \dots, T$, the learner

1. Chooses arm $i_t \in [K]$.
2. Receives reward $r_t \sim R_{i_t}$.

A popular objective for MAB is the pseudo-regret, which poses the *exploration-exploitation* challenge:

$$\text{Regret}_T = \sum_{t=1}^T (\mu^* - \mu_{i_t}).$$

Another important objective is the simple regret:

$$\mu^* - \mu_{\hat{i}},$$

where \hat{i} is the arm that the learner picks after T rounds of interactions. This poses the "pure exploration" challenge, since all it matters is to make a good final guess and the regret incurred within the T rounds does not matter. A related objective is called Best-Arm Identification, which asks whether $\hat{i} \in \arg \max_{i \in [K]} \mu_i$; Best-Arm Identification results often require additional gap conditions.

2.2 Uniform sampling

We consider the simplest algorithm that chooses each arm the same number of times, and after T rounds selects the arm with the highest empirical mean. For simplicity let's assume that T/K is an integer. We will prove a high-probability bound on the simple regret. The analysis gives an example of the application of Hoeffding's inequality to a learning problem; the algorithm itself is likely to be suboptimal.

For simplicity let's assume that T/K is an integer. After T rounds, each arm is chosen T/K times, and let $\hat{\mu}_i$ be the empirical average reward associated with arm i . By Hoeffding's inequality, we have:

$$\Pr[|\hat{\mu}_i - \mu_i| \geq \epsilon] \leq 2e^{-2T\epsilon^2/K}.$$

Now we want accurate estimation for *all* arms simultaneously. That is, we want to bound the probability of the event that *any* $\hat{\mu}_i$ deviating from μ_i too much. This is where union bound is useful:

$$\begin{aligned} & \Pr \left[\bigcup_{i=1}^K \{|\hat{\mu}_i - \mu_i| \geq \epsilon\} \right] && \text{(the event that estimation is } \epsilon\text{-inaccurate for at least 1 arm)} \\ & \leq \sum_{i=1}^K \Pr [|\hat{\mu}_i - \mu_i| \geq \epsilon] \leq 2K e^{-2T\epsilon^2/K}. && \text{(union bound, then Hoeffding's inequality)} \end{aligned}$$

To rephrase this result: with probability at least $1 - \delta$, $|\hat{\mu}_i - \mu_i| \leq \sqrt{\frac{K}{2T} \ln \frac{2K}{\delta}}$ holds for all i simultaneously.

Finally, we use the estimation error to bound the decision loss: recall that $\hat{i} = \arg \max_{i \in [K]} \hat{\mu}_i$, and let $i^* = \arg \max_{i \in [K]} \mu_i$.

$$\begin{aligned} \mu^* - \mu_{\hat{i}} &= \mu_{i^*} - \hat{\mu}_{i^*} + \hat{\mu}_{i^*} - \mu_{\hat{i}} \\ &\leq \mu_{i^*} - \hat{\mu}_{i^*} + \hat{\mu}_{\hat{i}} - \mu_{\hat{i}} \leq 2\sqrt{\frac{K}{2T} \ln \frac{2K}{\delta}}. \end{aligned}$$

We can rephrase this result as a sample complexity statement: in order to guarantee that $\mu^* - \mu_{\hat{i}} \leq \epsilon$ with probability at least $1 - \delta$, we need $T = O\left(\frac{K}{\epsilon^2} \ln \frac{K}{\delta}\right)$.

2.3 Lower bound

The linear dependence of the sample complexity on K makes a lot of sense, as to choose a arm with high reward we have to try each arm at least once. Below we will see how to mathematically formalize this idea and prove a lower bound on the sample complexity of MAB.

Theorem 2. *For any $K \geq 2$, $\epsilon \leq \sqrt{1/8}$, and any MAB algorithm, there exists an MAB instance where μ^* is ϵ better than other arms, yet the algorithm identifies the best arm with no more than $2/3$ probability unless $T \geq \frac{K}{72\epsilon^2}$.*

The theorem itself is stated as a best-arm identification lower bound, but it is also a lower bound for simple regret minimization. This is because all arms except the best one is ϵ worse than μ^* , so missing the optimal arm means a simple regret of at least ϵ .

See the proof in [1] (Theorem 2); the technique is due to [2] and can be also used to prove the lower bound on the regret of MAB.

3 Generalization Bounds for Supervised Learning

Consider a simple supervised learning setting: let \mathcal{X} be the feature space and \mathcal{Y} be the label space; in this example we consider classification so $\mathcal{Y} = \{0, 1\}$. Let $P_{X,Y}$ be a distribution over $\mathcal{X} \times \mathcal{Y}$, and we are given a dataset $\{(X_i, Y_i)\}_{i=1}^n$ with each (X_i, Y_i) drawn i.i.d. from $P_{X,Y}$. Let $\mathcal{F} : \mathcal{X} \rightarrow \mathcal{Y}$ be a finite hypothesis class. The classifier in \mathcal{F} that minimizes the classification error is:

$$f^* := \arg \min_{f \in \mathcal{F}} \mathbb{E}[\mathbb{I}[f(X) \neq Y]],$$

where $\mathbb{E}[\cdot]$ is w.r.t. $P_{X,Y}$. Given only a finite sample, one natural thing to do is *empirical risk minimization*, i.e., find the classifier that has the lowest training error rate on data:

$$\hat{f} = \arg \min_{f \in \mathcal{F}} \widehat{\mathbb{E}}[\mathbb{I}[f(X) \neq Y]] := \frac{1}{n} \sum_{i=1}^n \mathbb{I}[f(X_i) \neq Y_i].$$

The question is, can we give any guarantee to how good the learned classifier \hat{f} is compared to the optimal one f^* , as a function of n ? In other words, we want to bound

$$\mathbb{E}[\mathbb{I}[\hat{f}(X) \neq Y]] - \mathbb{E}[\mathbb{I}[f^*(X) \neq Y]].$$

We provide the analysis below, which mainly uses Hoeffding's and union bound. First of all,

$$\begin{aligned} & \mathbb{E}[\mathbb{I}[\hat{f}(X) \neq Y]] - \mathbb{E}[\mathbb{I}[f^*(X) \neq Y]] \\ & \leq \mathbb{E}[\mathbb{I}[\hat{f}(X) \neq Y]] - \widehat{\mathbb{E}}[\mathbb{I}[\hat{f}(X) \neq Y]] + \widehat{\mathbb{E}}[\mathbb{I}[f^*(X) \neq Y]] - \mathbb{E}[\mathbb{I}[f^*(X) \neq Y]] \quad (\hat{f} \text{ is optimal w.r.t. } \widehat{\mathbb{E}}) \\ & \leq 2 \cdot \max_{f \in \mathcal{F}} |\mathbb{E}[\mathbb{I}[f(X) \neq Y]] - \widehat{\mathbb{E}}[\mathbb{I}[f(X) \neq Y]]|. \end{aligned} \quad (3)$$

It then suffices to bound $\max_{f \in \mathcal{F}} |\mathbb{E}[\mathbb{I}[f(X) \neq Y]] - \widehat{\mathbb{E}}[\mathbb{I}[f(X) \neq Y]]|$, which is often called a *uniform deviation bound*. The key is to realize that, for any fixed $f \in \mathcal{F}$, $\widehat{\mathbb{E}}[\mathbb{I}[f(X) \neq Y]]$ is the average of i.i.d. random variables $\mathbb{I}[f(X_i) \neq Y_i]$ bounded in $[0, 1]$, whose true expectation is precisely $\mathbb{E}[\mathbb{I}[f(X) \neq Y]]$. Applying Hoeffding's, for a fixed $f \in \mathcal{F}$, with probability at least $1 - \delta$, we have

$$|\widehat{\mathbb{E}}[\mathbb{I}[f(X) \neq Y]] - \mathbb{E}[\mathbb{I}[f(X) \neq Y]]| \leq \sqrt{\frac{1}{2n} \ln \frac{2}{\delta}}.$$

Union bounding over \mathcal{F} and plugging into Eq.(4),

$$\mathbb{E}[\mathbb{I}[\hat{f}(X) \neq Y]] - \mathbb{E}[\mathbb{I}[f^*(X) \neq Y]] \leq \sqrt{\frac{2}{n} \ln \frac{2|\mathcal{F}|}{\delta}}. \quad (4)$$

References

- [1] Akshay Krishnamurthy, Alekh Agarwal, and John Langford. PAC reinforcement learning with rich observations. In *Advances in Neural Information Processing Systems*, pages 1840–1848, 2016.
- [2] Peter Auer, Nicolò Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine learning*, 47(2-3):235–256, 2002.