# Concentration ineq. & union bound.

Let $X_1, X_2, \dots, X_n$ be iid. r.v.

we know: $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{n\to\infty} \mathbb{E}[X_1]$.

Hoeffding's: if $X_1, \dots, X_n$ are bounded in $[a, b]$ almost surely.

with probability at least $1-\delta$.

$$\Big| \underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\text{emp. avg}} - \underbrace{\mathbb{E}[X_1]}_{\text{true mean}} \Big| \le (b-a)\cdot\sqrt{\frac{1}{2n}\ln\frac{2}{\delta}}.$$

(Alternatively: $\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}[X_1]\right| > (b-a)\sqrt{\frac{1}{2n}\ln\frac{2}{\delta}}\right] \le \delta$.).

---

Union bound: given events $A_1, A_2, \dots, A_m$

$$\mathbb{P}\left[\bigcup_{i=1}^{m} A_i\right] \le \sum_{i=1}^{m} \mathbb{P}[A_i].$$

## Multi-armed Bandits

$A = \{1, 2, 3, \cdots, K\}$.

For $t = 1, 2, 3, \cdots, T$.

1. Learner chooses arm $i_t \in [K]$.
2. receives random reward $r_t \sim R_{i_t}$.

where $\forall i$, $R_i$ is the reward distribution.
w/ bounded range $[0, 1]$. & mean $\mu_i$.

Define $\mu^* = \max_i \mu_i$. & $i^* = \arg\max_i \mu_i$.

Typical objectives:

(1) regret: $\sum_{t=1}^{T} (\mu^* - \mu_{i_t}) = o(T)$.

$\boxed{(2)}$ best-arm identification: after $T$ rounds.
algorithm outputs $\hat{i} \in [K]$.
obj: $\mu^* - \mu_{\hat{i}} \leq O\left(\sqrt{\frac{1}{T}}\right)$.

---

Algorithm: sample each arm $T/K$ times.
↳ assumes integer.

& let $\hat{\mu}_i$ be the emp. avg. reward.

$\Rightarrow \hat{i} = \arg\max_i \hat{\mu}_i$.

Analysis: $\begin{cases} 1. \text{ show that } |\mu_i - \hat{\mu}_i| \text{ is small for all } i. \\ 2. \text{ translate } \max_i |\mu_i - \hat{\mu}_i| \longrightarrow \mu^* - \mu_i. \end{cases}$

Step 1: Fix arbitrary $i \in [K]$. by Hoeffding's.

$$|\mu_i - \hat{\mu}_i| \le \sqrt{\frac{1}{2n} \ln \frac{2}{\delta}} \quad w.p. \ge 1-\delta.$$

$\Rightarrow$ w.p. $\ge 1-\delta$. $\max_i |\mu_i - \hat{\mu}_i| \le \sqrt{\frac{1}{2n} \ln \frac{2K}{\delta}}$

Proof: $\forall i$, $\mathbb{P}\left[|\mu_i - \hat{\mu}_i| > \sqrt{\frac{1}{2n} \ln \frac{2}{\delta'}}\right] < \delta'$.

$$\Rightarrow \mathbb{P}\left[\max_i |\mu_i - \hat{\mu}_i| > ?\right] < \delta.$$

$$\mathbb{P}\left[\bigcup_{i=1}^{K} \{|\mu_i - \hat{\mu}_i| > ?\}\right]$$

$$\le \sum_{i=1}^{K} \mathbb{P}\left[|\mu_i - \hat{\mu}_i| > ?\right] \xrightarrow{\text{we want to guarantee}} < \underline{\delta}.$$

it suffices to have $\forall i$, $\mathbb{P}\left[|\mu_i - \hat{\mu}_i| > ?\right] < \frac{\delta}{K}$.

$$\Rightarrow ? = \sqrt{\frac{1}{2n} \ln \frac{2}{\delta/K}}.$$

$$\mathbb{P}\left[|\mu_1 - \hat{\mu}_1| \le ?, |\mu_2 - \hat{\mu}_2| \le ?, \dots\right]$$

$$= \underline{\underline{(1-\delta')^K}} \approx \underbrace{1 - K \cdot \delta'}_{\text{union bound}}.$$

Step 2. $\mu^* - \mu_{\hat{i}}$ "decision loss"

$$\le \underbrace{\mu_{i^*} - \hat{\mu}_{i^*}} + \underbrace{\hat{\mu}_{\hat{i}} - \mu_{\hat{i}}} \quad (\hat{\mu}_{\hat{i}} \ge \hat{\mu}_i, \forall i)$$

$$\le 2 \cdot \max_i |\mu_i - \hat{\mu}_i| \quad \text{"evaluation error"}.$$

$$\le 2 \cdot \sqrt{\frac{1}{2n} \ln \frac{2K}{\delta}}, \quad \text{w.p.} \ge 1-\delta. \quad \square$$

---

## Example 2: Generalization error bounds

$\mathcal{X}$ feature space, $\mathcal{Y} = \{0,1\}$ label space.

Data: $\{(X_i, Y_i)\}_{i=1}^n \overset{iid}{\sim} P_{X,Y}$.

Hypothesis class: $\mathcal{F} \subseteq (\mathcal{X} \to \mathcal{Y})$. (assume finite $\mathcal{F}$).

Algorithm: $\hat{f} = \operatorname{argmin}_{f \in \mathcal{F}} \underbrace{\frac{1}{n} \sum_{i=1}^n \underbrace{\mathbb{I}[f(X_i) \ne Y_i]}_{\text{misclassification}}}_{\hat{\mathbb{E}}[\mathbb{I}[f(X) \ne Y]]}$

Goal:

best prediction: $f^* := \operatorname{argmin}_{f \in \mathcal{F}} \underbrace{\mathbb{E}[\mathbb{I}(f(X) \ne Y)]}$.

can we bound $\mathbb{E}[\mathbb{I}(\hat{f}(X) \ne Y)] - \mathbb{E}[\mathbb{I}(f^*(X) \ne Y)]$

Analysis. $\begin{cases} 1. & \max_{f \in \mathcal{F}} \left| \hat{\mathbb{E}}[\mathbb{I}(f(X) \neq Y)] - \mathbb{E}[\mathbb{I}(f(X) \neq Y)] \right| \\ 2. & \text{use this to bound the final obj.} \end{cases}$

Step 1: $\hat{\mathbb{E}}[\mathbb{I}(f(X) \neq Y)] = \frac{1}{n} \sum_{i=1}^{n} \underbrace{\mathbb{I}[f(X_i) \neq Y_i]}$ iid. r.v. $\in [0,1]$.

$\boxed{\forall \text{ (fixed) } f. \ \text{w.p.} \geq 1-\delta}$ (Hoeffding's.)

$$\underline{\left| \hat{\mathbb{E}}[\mathbb{I}(f(X) \neq Y)] - \mathbb{E}[\cdots] \right|} \leq \sqrt{\frac{1}{2n} \ln \frac{2}{\delta}}.$$

$\Rightarrow$ Therefore, we have $\boxed{\text{w.p.} \geq 1-\delta, \ \forall f \in \mathcal{F} \text{ (simultaneously)}}$

$$\left| \hat{\mathbb{E}}[\mathbb{I}(f(X) \neq Y)] - \mathbb{E}[\cdots] \right| \leq \sqrt{\frac{1}{2n} \ln \frac{2|\mathcal{F}|}{\delta}}.$$

Remarks: 1. estimation events are not independent. but union bound still applies.

2. order of "$\forall f$" & "w.p." matters.

Step 2: $\mathbb{E}[\mathbb{I}(\hat{f}(X) \neq Y)] - \mathbb{E}[\mathbb{I}(f^*(X) \neq Y)]$

$$\leq \quad - \hat{\mathbb{E}}[\cdots] + \hat{\mathbb{E}}[\cdots]$$

$$\leq 2 \cdot \max_{f \in \mathcal{F}} \left| \hat{\mathbb{E}}[\cdots] - \mathbb{E}[\cdots] \right|$$

$$\leq 2 \cdot \underline{\sqrt{\frac{1}{2n} \ln \frac{2|\mathcal{F}|}{\delta}}} \quad \text{w.p.} \geq 1-\delta.$$

Remark: the bound depends on $\log|\mathcal{F}|$.

$\Rightarrow$ can handle exponentially large $|\mathcal{F}|$.

e.g. Linear regression

$f(X) = \theta^T \phi(X)$

$\mathcal{F} = \{\theta^T \phi : \|\theta\|_\infty \leq C\}$

discretize.

$\Rightarrow$ # functions $= \left(\frac{2C}{\varepsilon}\right)^d$

$\log\left(\frac{2C}{\varepsilon}\right)^d \approx d$

"covering number".