Policy Gradient
(ref: notes on course website; not all contents in notes are covered in class)
Policy Gradient (PG)

• Given a class of parameterized policies $\pi_\theta$, optimize
  
  $v^{\pi_\theta} := \mathbb{E}_{s \sim \mu}[V^{\pi_\theta}(s)]$

• We will often make the dependence of $\pi_\theta$ on $\theta$ implicit, i.e., when we write $\pi$ we mean $\pi_\theta$ in this part of the course

• Simple idea: can run (stochastic) gradient descent if we can obtain (an unbiased estimate of) $\nabla_\theta v^{\pi_\theta}$

• will abbreviate as $\nabla v^{\pi}$

• Beautiful result: an unbiased estimate can be obtained from a single on-policy trajectory, without using knowledge of $P$ and $R$ of the MDP!

• Has a strong connection to IS

• “Vanilla” PG (e.g., REINFORCE) is considered a Monte-Carlo method—it does not leverage Bellman equation
Why PG?

- RL methods can be categorized according to what we try to approximate: model-based RL, value-based RL, policy search
- Eventually we only care about a good policy!
- value-based RL is indirect (model-based even more)
- If a value function induces a good greedy policy, but the function itself severely violates Bellman equation, you won’t be able to find such a policy via value-based methods
- In other words, policy search is agnostic against misspecification of function approximation
  - Apart from difficulties in optimization, there is nothing that prevents policy search from finding the best policy in class
- Value- (and model-) based methods have their advantages—will come back later
Example of policy parametrization

• Linear + softmax:
  • Featurize state-action: $\phi : S \times A \rightarrow \mathbb{R}^d$
  • Policy: $\pi(a \mid s) \propto e^{\theta^T \phi(s,a)}$

• Recall that in SARSA we’ve also seen the softmax policy
• There we include a temperature parameter, $\pi(a \mid s) \propto e^{\theta^T \phi(s,a)/T}$
• Why the difference?
  • In TD, we want $\theta^T \phi(s, a) \approx Q^\pi(s, a)$. We don’t have the freedom to rescale it; i.e., if $\theta^T \phi(s, a) \approx Q^\pi(s, a)$, then $(2\theta)^T \phi(s, a) \neq Q^\pi(s, a)$.
  • We need an additional knob ($T$) to control the stochasticity of $\pi$
  • In PG, $\theta^T \phi(s, a)$ does not carry any meaning—it’s totally possible that eventually we find a $\theta$ but $\theta^T \phi(s, a) \neq Q^\pi_{\theta}(s, a)$!
  • That’s why we can absorb the temperature parameter in $\theta$
  • Reflection of the agnosticism of PG
Derivation of PG

- Use $\tau := (s_1, a_1, r_1, \ldots, s_H, a_H, r_H)$ to denote a trajectory (episodic)
- Use $\tau \sim \pi$ as a shorthand for distribution induced by $\pi$
- Let $R(\tau) := \sum_{t=1}^H \gamma^{t-1} r_t$
- **Ver 1:** $\nabla v^\pi = \mathbb{E}_{\tau \sim \pi}[R(\tau) \sum_{t=1}^H \nabla \log \pi(a_t | s_t)]$
  - Will derive using a “MC”-style proof
- **Ver 2:** $\nabla v^\pi = \frac{1}{1-\gamma} \mathbb{E}_{s \sim \eta^\pi, a \sim \pi(s)}[Q^\pi(s, a) \nabla \log \pi(a | s)]$
  - $\eta^\pi$ is the normalized occupancy (from $\mu$ as init distribution)
  - Possible implementation: (1) roll out $\tau \sim \pi$, (2) pick a random time step $t$ w.p. $\propto \gamma^{t-1}$, (3) $(\sum_{t'=t}^H \gamma^{t'-1} r_{t'}) \nabla \log \pi(a_t | s_t)$
    - Note that $\mathbb{E}[\sum_{t'=t}^H \gamma^{t'-1} r_{t'} | s_t, a_t] = Q^\pi(s_t, a_t)$
    - Take expectation over step (2) gives an alternative form:
      $\nabla v^\pi = \mathbb{E}_{\tau \sim \pi}[\sum_{t=1}^H (\sum_{t'=t}^H \gamma^{t'-1} r_{t'}) \nabla \log \pi(a_t | s_t)]$
  - Will derive using a “DP”-style proof; can also be derived using the MC-style proof for ver 1
Pros & Cons of PG, and beyond

- Standard PG is fully on-policy, and it’s hard to reuse data
  - after each update step, the policy changes and we need to generate MC trajectories from the new policy
- in practice, it suffers from noisy gradient estimate
- Blend PG with value-based method:
  - \[ \nabla v^\pi = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \eta, a \sim \pi(s)}[Q^\pi(s, a) \nabla \log \pi(a \mid s)] \]
  - Instead of using MC estimate \( \sum_{t' = t}^{H} \gamma^{t' - 1} r_t \) for \( Q^\pi(s_t, a_t) \), use an approximate value-function \( \hat{Q}^\pi(s_t, a_t) \), often trained by TD
  - e.g., using expected Sarsa—can leverage previous (off-policy) data to learn \( \hat{Q}^\pi(s_t, a_t) \)
  - “Actor-critic”: the parametrized policy is called the actor, and the value-function estimate is called the critic
Baseline in PG

- \[ \nabla v^\pi = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \eta, a \sim \pi(s)}[Q^\pi(s, a) \nabla \log \pi(a \mid s)] \]
- For any \( f : S \to \mathbb{R}, \ \nabla v^\pi = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \eta, a \sim \pi(s)}[(Q^\pi(s, a) - f(s)) \nabla \log \pi(a \mid s)] \]
  - for any \( s \), \( \mathbb{E}_{a \sim \pi(s)}[f(s) \nabla \log \pi(a \mid s)] = f(s) \cdot \mathbb{E}_{a \sim \pi(s)}[\nabla \log \pi(a \mid s)] = 0 \)
  - proof: \( \mathbb{E}_{a \sim \pi(s)}[\nabla \log \pi(a \mid s)] = \sum_a \pi(a \mid s) \nabla \log \pi(a \mid s) = \nabla \sum_a \pi(a \mid s) = \nabla 1 = 0 \)
- One choice: \( f = V^\pi(s) \)
  - \[ \nabla v^\pi = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \eta, a \sim \pi(s)}[A^\pi(s, a) \nabla \log \pi(a \mid s)] \]
  - recall that \( A \) is the advantage function
Comparing AC with Policy Iteration

- \[ \nabla v^\pi = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \eta^\pi, a \sim \pi(s)}[\hat{Q}^\pi(s, a) \nabla \log \pi(a | s)] \]

- A different but related procedure: freeze \( \pi \), update the parameter of another policy \( \pi' \) (whose parameters are \( \theta' \)) by
  \[ \theta' \leftarrow \theta' + \alpha \cdot \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \eta^\pi, a \sim \pi(s)}[\hat{Q}^\pi(s, a) \nabla \log \pi'(a | s)] \]

- gradient = 0 at \( \pi' = \pi_{Q^\pi} \) => policy iteration

- This can run into serious issues
  - Tabular PI theory assumes that we get \( \hat{Q}^\pi \) that is accurate for every single state-action pair
  - Simply unrealistic if problem is complex and we can only rollout trajectories (instead of sweeping the entire state space)
  - in the middle of learning, part of the state space may be under-explored
  - at best we can hope \( \hat{Q}^\pi \) to be accurate under distribution of state space we have data for
Comparing AC with Policy Iteration

• \( \nabla v^\pi = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim \eta^\pi, a \sim \pi(s)}[\hat{Q}^\pi(s, a) \nabla \log \pi(a \mid s)] \)

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  \]
• gradient = 0 at \( \pi' = \pi_{Q^\pi} \) => policy iteration

• This can run into serious issues
  • (cont.) if \( \pi' \) visits new states, \( \hat{Q}^\pi \) may be highly inaccurate in those states, and policy improvement no longer holds
  • Perhaps better idea: move \( \pi' \) a little more but not too far from \( \pi \), so that their state occupancies are still similar.

• Theory: CPI [Kakade & Langford’02]
• Modern implementations & variants: TRPO, PPO, etc
RL Algorithms Landscape

- policy search
- value-based RL

0-th order opt.