Policy Gradient

Policy Gradient (PG)

- Given a class of parameterized policies π_θ, optimize $J(\pi_\theta) := \mathbb{E}_{s \sim d_0}[V^{\pi_\theta}(s)]$
  - We will often make the dependence of π_θ on θ implicit, i.e., when we write π we mean π_θ in this part of the course
- Simple idea: can run (stochastic) gradient descent if we can obtain (an unbiased estimate of) $\nabla_\theta J(\pi_\theta)$
  - will abbreviate as $J(\pi)$
- Beautiful result: an unbiased estimate can be obtained from a single on-policy trajectory, without using knowledge of P and R of the MDP!
- Has a strong connection to IS
- "Vanilla" PG (e.g., REINFORCE) is considered a Monte-Carlo method—it does not leverage Bellman equation

Why PG?

- RL methods can be categorized according to what we try to approximate: model-based RL, value-based RL, policy search
- Eventually we only care about a good policy!
- value-based RL is indirect (model-based even more)
- If a value function induces a good greedy policy, but the function itself severely violates Bellman equation, you won't be able to find such a policy via value-based methods
- In other words, policy search is agnostic against misspecification of function approximation
    - Apart from difficulties in optimization, there is nothing that prevents policy search from finding the best policy in class
- Value- (and model-) based methods have their advantages—will come back later

Example of policy parametrization

- Linear + softmax:
  - Featurize state-action: $\phi : S \times A \to \mathbb{R}^d$
  - Policy: $\pi(a \mid s) \propto e^{\theta^\top \phi(s,a)}$
- Recall that in SARSA we've also seen the softmax policy
- There we include a temperature parameter, $\pi(a \mid s) \propto e^{\theta^\top \phi(s,a)/T}$
- Why the difference?
  - In TD, we want $\theta^\top \phi(s, a) \approx Q^\pi(s, a)$. We don't have the freedom to rescale it; i.e., if $\theta^\top \phi(s, a) \approx Q^\pi(s, a)$, then $(2\theta)^\top \phi(s, a) \neq Q^\pi(s, a)$.
  - We need an additional knob (T) to control the stochasticity of π
  - In PG, $\theta^\top \phi(s, a)$ does not carry any meaning—it's totally possible that eventually we find a θ but $\theta^\top \phi(s, a) \neq Q^{\pi_\theta}(s, a)$!
  - That's why we can absorb the temperature parameter in θ
  - Reflection of the agnosticity of PG

Derivation of PG

- Use $\tau := (s_1, a_1, r_1, \ldots, s_H, a_H, r_H)$ to denote a trajectory (episodic)
- Use $\tau \sim \pi$ as a shorthand for distribution induced by π
- Let $R(\tau) := \sum_{t=1}^{H} \gamma^{t-1} r_t$
- Ver 1: $\nabla J(\pi) = \mathbb{E}_{\tau\sim\pi}[R(\tau) \sum_{t=1}^{H} \nabla \log \pi(a_t \mid s_t)]$ "REINFORCE"
  - Will derive using a "MC"-style proof
- Ver 2: $\nabla J(\pi) = \frac{1}{1-\gamma} \mathbb{E}_{s\sim d^\pi, a\sim\pi(s)}[Q^\pi(s,a) \nabla \log \pi(a \mid s)]$
  - d^π is the normalized occupancy (from d_0 as init distribution)
  - Possible implementation: (1) roll out $\tau \sim \pi$, (2) pick a random time step t w.p. $\propto \gamma^{t-1}$, (3) $(\sum_{t'=t}^{H} \gamma^{t'-1} r_t) \nabla \log \pi(a_t \mid s_t)$
    - Note that $\mathbb{E}[\sum_{t'=t}^{H} \gamma^{t'-1} r_t \mid s_t, a_t] = Q^\pi(s_t, a_t)$
    - Take expectation over step (2) gives an alternative form: $\nabla J(\pi) = \mathbb{E}_{\tau\sim\pi}[\sum_{t=1}^{H} (\sum_{t'=t}^{H} \gamma^{t'-1} r_{t'}) \nabla \log \pi(a_t \mid s_t)]$
  - Will derive using a "DP"-style proof; can also be derived using the MC-style proof for ver 1

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$$J(\pi) = \mathbb{E}_\pi\left[\sum_{t=1}^{H} \gamma^{t-1} r_t\right] = \mathbb{E}_{\tau\sim\pi}[R(\tau)].$$

$$\frac{d \log y}{dx} = \frac{1}{y}\frac{dy}{dx}.$$

$$\nabla J(\pi) \quad (= \nabla_\theta J(\pi_\theta)).$$

$$= \nabla \left(\sum_{\tau \in (S\times A)^H} P^\pi(\tau) R(\tau) \right)$$

$$= \sum_\tau \Big(\nabla_\theta \underbrace{P^{\pi_\theta}(\tau)}_{y}\Big) R(\tau).$$

$$= \sum_\tau P^\pi(\tau) \left(\nabla_\theta \log P^{\pi_\theta}(\tau) \right) \cdot R(\tau).$$

$$= \mathbb{E}_{\tau\sim\pi}\left[\nabla_\theta \log P^{\pi_\theta}(\tau) \cdot \boxed{R(\tau)} \right]$$

$$\nabla_\theta \log P^{\pi_\theta}(\tau). \qquad s_1, a_1, s_2, a_2, \ldots,$$

$$= \nabla_\theta \log \left(d_0(s_1) \cdot \pi(a_1|s_1) P(s_2|s_1,a_1) \pi(a_2|s_2) \cdots \right)$$

$$= \cancel{\nabla \log d_0(s_1)} + \nabla \log \pi(a_1|s_1) + \cancel{\nabla \log P(s_2|s_1,a_1)} + \nabla \log \pi(a_2|s_2) + \cdots).$$

$$\nabla \log \boxed{\pi_\theta(a|s) = \frac{e^{\theta^T \phi(s,a)}}{\sum_{a'} e^{\theta^T \phi(s,a')}}}$$

$$= \nabla_\theta \left(\underline{\log\left(e^{\theta^T\phi(s,a)}\right)} - \underline{\log\left(\sum_{a'} e^{\theta^T \phi(s,a')}\right)} \right)$$

$$= \phi(s,a) - \boxed{\frac{\sum_{a'} e^{\theta^T\phi(s,a')} \cdot \phi(s,a')}{\sum_{a'} e^{\theta^T\phi(s,a')}}}$$

$$= \underbrace{\phi(s,a) - \mathbb{E}_{a'\sim\pi}\left[\phi(s,a')\right]}.$$

$$\mathbb{E}_{a\sim\pi}[\,\cdot\,] = ?$$

$$\nabla J(\pi) = \mathbb{E}_\pi\left[\left(\sum_{t=1}^{H} \gamma^{t-1} r_t\right)\left(\sum_{t=1}^{H} \nabla \log \pi(a_t|s_t)\right)\right]$$

$$= \mathbb{E}_\pi\left[\sum_{t=1}^{H}\left(\nabla \log \pi(a_t|s_t) \sum_{t'=1}^{H} \gamma^{t'-1} r_{t'}\right)\right]$$

$$= \mathbb{E}_\pi\left[\sum_{t=1}^{H} \nabla \log \pi(a_t|s_t) \sum_{t'=t}^{H} \gamma^{t'-1} r_{t'}\right].$$

$$\frac{dJ(\pi)}{d\theta} = \lim_{\Delta\theta \to 0} \frac{J(\pi_{\theta+\Delta\theta}) - J(\pi_\theta)}{\Delta\theta}$$

IS MC

$$\mathbb{E}_{\pi}\left[\sum_{t=1}^{H} \nabla \log \pi(a_t|s_t) \sum_{t'=t}^{H} \gamma^{t'-1} r_{t'}\right]$$

$$= \mathbb{E}_{\substack{s_t,a_t \\ \sim\pi}}\left[\mathbb{E}_{\pi}\left[\sum_{t=1}^{H} \nabla \log \pi(a_t|s_t) \sum_{t'=t}^{H} \gamma^{t'-1} r_{t'} \,\middle|\, s_t, a_t\right]\right]$$

$$\sum_{\boxed{t=1}}^{H} \mathbb{E}_{s_t,a_t \sim d_t^\pi}\left[\gamma^{t-1} \cdot \nabla \log \pi(a_t|s_t)\, \underbrace{\mathbb{E}_{\pi}\left[\sum_{t'=t}^{H} \gamma^{t'-t} r_{t'} \,\middle|\, s_t, a_t\right]}_{Q^\pi(s_t,a_t)}\right]$$

$$= \sum_{t=1}^{H} \gamma^{t-1} \mathbb{E}_{s,a\sim d_t^\pi}\left[\nabla \log \pi(a|s) \cdot Q^\pi(s,a)\right].$$

$$= \frac{1}{1-\gamma} \mathbb{E}_{\underbrace{(1-\gamma)\cdot\sum_{t=1}^{H} d_t^\pi}_{d^\pi}}\left[\nabla \log \pi(a|s) \cdot Q^\pi(s,a)\right].$$

$$\mathbb{E}_p[f] + \mathbb{E}_q[f] = 2 \cdot \mathbb{E}_{\frac{p+q}{2}}[f].$$

$a = L, \; a = R.$

$$\boxed{\begin{aligned} Q^*(s,L) &\approx \theta_L^T \psi(s) \\ Q^*(s,R) &\simeq \theta_R^T \psi(s) \end{aligned}} \in \mathbb{R}^d.$$

$$Q^*(s,a) = \theta^T \phi(s,a).$$

$$= \begin{bmatrix} \theta_L \\ \theta_R \end{bmatrix}^T \phi(s,a)$$

$$\phi(s,L) = \begin{bmatrix} \psi(s) \\ 0 \end{bmatrix} \qquad \phi(s,R) = \begin{bmatrix} 0 \\ \psi(s) \end{bmatrix}.$$

$$\begin{bmatrix} \theta_L \\ \theta_R \end{bmatrix} \leftarrow (\;TD \cdots\;) \cdot \begin{bmatrix} \psi(s) \\ 0 \end{bmatrix},$$

$s_t = s, \; a_t = L.$

Pros & Cons of PG, and beyond

- Standard PG is fully on-policy, and it's hard to reuse data
  - after each update step, the policy changes and we need to generate MC trajectories from the new policy
- in practice, it suffers from noisy gradient estimate
- Blend PG with value-based method:
  - $\nabla J(\pi) = \frac{1}{1-\gamma} \mathbb{E}_{s \sim d^{\pi}, a \sim \pi(s)} [Q^{\pi}(s, a) \nabla \log \pi(a \mid s)]$
  - Instead of using MC estimate $\sum_{t'=t}^{H} \gamma^{t'-1} r_t$ for $Q^{\pi}(s_t, a_t)$, use an approximate value-function $\hat{Q}^{\pi}(s_t, a_t)$, often trained by TD
  - e.g., using expected Sarsa—can leverage previous (off-policy) data to learn $\hat{Q}^{\pi}(s_t, a_t)$
  - "Actor-critic": the parametrized policy is called the actor, and the value-function estimate is called the critic

Baseline in PG

- $\nabla J(\pi) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^\pi, a\sim\pi(s)}[Q^\pi(s,a)\,\nabla\log\pi(a\,|\,s)]$
- For any $f: S \to \mathbb{R}$, $\nabla J(\pi) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^\pi, a\sim\pi(s)}[(Q^\pi(s,a) - f(s))\,\nabla\log\pi(a\,|\,s)]$
  - for any s, $\mathbb{E}_{a\sim\pi(s)}[f(s)\,\nabla\log\pi(a\,|\,s)] = f(s)\cdot\mathbb{E}_{a\sim\pi(s)}[\nabla\log\pi(a\,|\,s)] = 0$
  - proof: $\mathbb{E}_{a\sim\pi(s)}[\nabla\log\pi(a\,|\,s)] = \sum_a \pi(a\,|\,s)\,\nabla\log\pi(a\,|\,s)$ $= \sum_a \nabla\pi(a\,|\,s) = \nabla\sum_a \pi(a\,|\,s) = \nabla 1 = 0$

$\frac{\nabla\pi(a|s)}{\pi(a|s)}$

- One choice: $f = V^\pi(s)$

$Q^\pi(s,a) - V^\pi(s)$

  - $\nabla J(\pi) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^\pi, a\sim\pi(s)}[A^\pi(s,a)\,\nabla\log\pi(a\,|\,s)]$
  - recall that A is the advantage function

Comparing AC with Policy Iteration

- $\nabla J(\pi) \approx \frac{1}{1-\gamma} \mathbb{E}_{s \sim d^{\pi}, a \sim \pi(s)} [\hat{Q}^{\pi}(s, a) \nabla \log \pi(a \mid s)]$
- A different but related procedure: freeze π, update the parameter of another policy π' (whose parameters are θ') by
  $\theta' \leftarrow \theta' + \alpha \cdot \frac{1}{1-\gamma} \mathbb{E}_{s \sim d^{\pi}, a \sim \pi(s)} [\hat{Q}^{\pi}(s, a) \nabla \log \pi'(a \mid s)]$
  - gradient = $\mathbf{0}$ at $\pi' = \pi_{Q^{\pi}}$ => policy iteration
- This can run into serious issues
  - Tabular PI theory assumes that we get $\hat{Q}^{\pi}$ that is accurate for every single state-action pair
  - Simply unrealistic if problem is complex and we can only roll-out trajectories (instead of sweeping the entire state space)
  - in the middle of learning, part of the state space may be under-explored
  - at best we can hope $\hat{Q}^{\pi}$ to be accurate under distribution of state space we have data for

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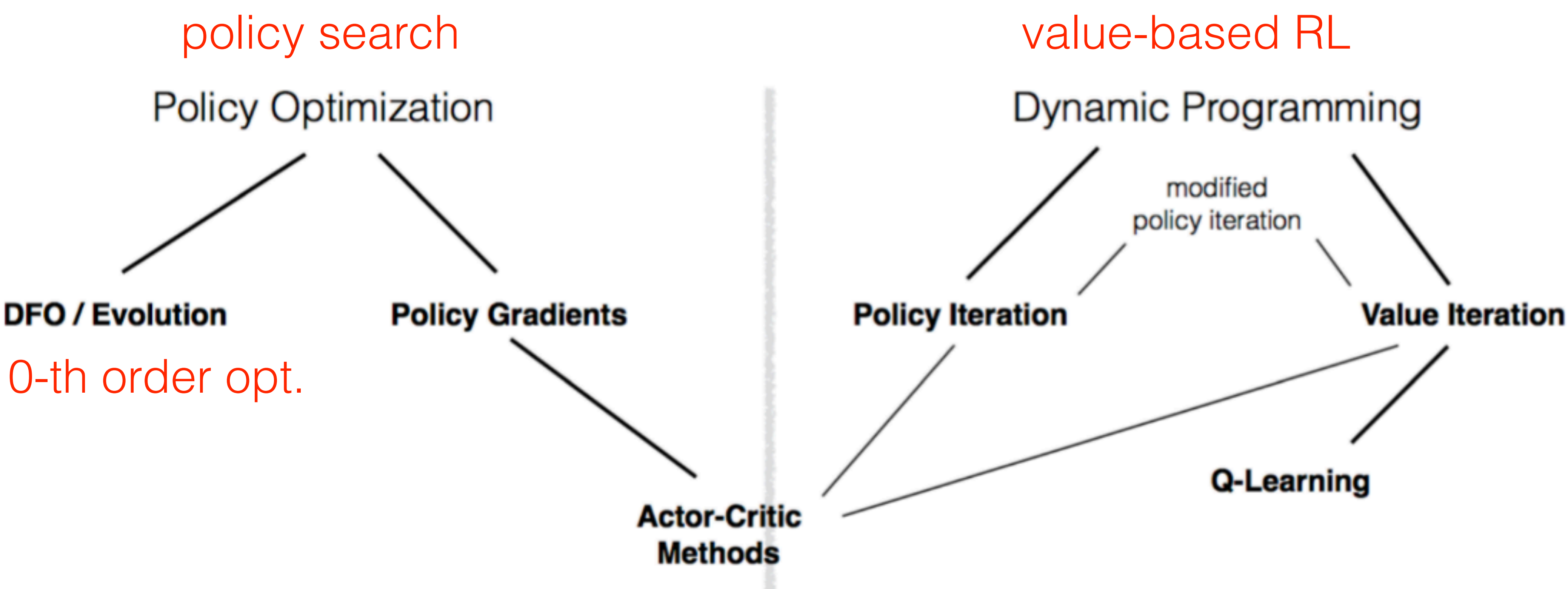
  - gradient = $\mathbf{0}$ at $\pi' = \pi_{Q^{\pi}}$ => policy iteration
- This can run into serious issues
  - (cont.) if π' visits new states, $\hat{Q}^{\pi}$ may be highly inaccurate in those states, and policy improvement no longer holds
- Perhaps better idea: move π' a little more but not too far from π, so that their state occupancies are still similar.
- Theory: CPI [Kakade & Langford'02]
- Modern implementations & variants: TRPO, PPO, etc

RL Algorithms Landscape



Slide Credit: Pieter Abbeel

Practical considerations

- Recall that one way to implement PG/AC is:
  1. roll out $\tau \sim \pi$,
  2. gradient from step *t*: $Q^{\pi}(s_t, a_t)\, \nabla \log \pi(a_t \mid s_t)$
  3. sum up the gradients from all time steps, with weight $\propto \gamma^{t-1}$,
- What if a trajectory length >> $1/(1-\gamma)$?
  - Most of the data points are wasted!
- Deep RL implementation in Atari games:
  - Trajectory length = ~5 min
  - Effective horizon = secs

$\gamma = 0.99$, frame rate 60Hz ⇨ effective horizon = $O(\,1/(1-\gamma) * 1/60)\,) = \sim$ sec

Practical considerations

- Actual implementation:
  1. roll out $\tau \sim \pi$,
  2. gradient from step t: $Q^{\pi}(s_t, a_t)\, \nabla \log \pi(a_t \mid s_t)$
  3. put equal weights on gradients from all time steps
- Pro: use all data points; Con: biased gradient.
- Is there no discounting then?
  - $Q^{\pi}(s_t, a_t)$ is still learned using γ (e.g., by TD in actor-critic)
- How to understand/make sense of this?